

A Linear-Quadratic Optimal Control Problem for Mean-Field Stochastic Differential Equations in Infinite Horizon*

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Abstract

A linear-quadratic (LQ, for short) optimal control problem is considered for mean-field stochastic differential equations with constant coefficients in an infinite horizon. The stabilizability of the control system is studied followed by the discussion of the well-posedness of the LQ problem. The optimal control can be expressed as a linear state feedback involving the state and its mean, through the solutions of two algebraic Riccati equations. The solvability of such kind of Riccati equations is investigated by means of semi-definite programming method.

Keywords. Mean-field stochastic differential equation, linear-quadratic optimal control, MF-stabilizability, Riccati equation.

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1 Introduction.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a complete filtered probability space, on which a one-dimensional standard Brownian motion $W(\cdot)$ is defined with $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t \geq 0}$ being its natural filtration augmented by all the \mathbb{P} -null sets. Consider the following controlled linear stochastic differential equation (SDE, for short) in \mathbb{R}^n :

$$\begin{cases} dX(t) = \left\{ AX(t) + \bar{A}\mathbb{E}[X(t)] + Bu(t) + \bar{B}\mathbb{E}[u(t)] \right\} dt \\ \quad + \left\{ CX(t) + \bar{C}\mathbb{E}[X(t)] + Du(t) + \bar{D}\mathbb{E}[u(t)] \right\} dW(t), \quad t \geq 0, \\ X(0) = x, \end{cases} \quad (1.1)$$

where $A, \bar{A}, C, \bar{C} \in \mathbb{R}^{n \times n}$ and $B, \bar{B}, D, \bar{D} \in \mathbb{R}^{n \times m}$ are given (deterministic) matrices. In the above, $X(\cdot)$, valued in \mathbb{R}^n , is called the *state process*, and $u(\cdot)$, valued in \mathbb{R}^m , is called a *control process*.

Different from classical controlled linear SDEs, the terms $\mathbb{E}[X(\cdot)]$ and $\mathbb{E}[u(\cdot)]$ appear in the equation. We call (1.1) a controlled mean-field (forward) SDE (MF-FSDE, for short). Historically, a special case of MF-FSDE, called McKean–Vlasov SDE, was suggested by Kac [23] in 1956 as a stochastic toy model for the Vlasov type kinetic equation of plasma and the rigorous study of which was initiated by McKean [28] in 1966. Since then, such kind of equations were studied by many authors, see, for examples, Dawson [17], Dawson–Gärtner [18], Gärtner [20], Scheutzow [32], Graham [21], Chan [14], Chiang [15], Ahmed–Ding [2], and the

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references cited therein. For the relevant works of recent years, see, for examples, Veretennikov [34], Huang–Malhamé–Caines [22], Buckdahn–Djehiche–Li–Peng [12], Buckdahn–Li–Peng [13], Borkar–Kumar [9], Crisan–Xiong [16], Kotelenez–Kurtz [25], and so on. Control problems of McKean–Vlasov equation or MF-FSDEs were investigated by Ahmed–Ding [3], Ahmed [4], Buckdahn–Djehiche–Li [11], Park–Balasubramanian–Kang [30], Andersson–Djehiche [6], Meyer–Brandis–Oksendal–Zhou [29], and so on. In Yong [35], a linear-quadratic (LQ, for short) problem was introduced and investigated for MF-FSDEs in finite horizons. Some interesting motivation was given in [35] for the control problem with $\mathbb{E}[X(\cdot)]$ and $\mathbb{E}[u(\cdot)]$ being included in the cost functional. This paper can be regarded as a continuation of [35], for LQ problem of MF-FSDEs in an infinite horizon.

We introduce the following:

$$\begin{cases} \mathcal{U}[0, T] = \left\{ u : [0, \infty) \times \Omega \rightarrow \mathbb{R}^m \mid u(\cdot) \text{ is } \mathbb{F}\text{-adapted, } \mathbb{E} \int_0^T |u(s)|^2 ds < \infty \right\}, & \forall T > 0, \\ \mathcal{U}_{loc}[0, \infty) = \bigcup_{T>0} \mathcal{U}[0, T], \\ \mathcal{U}[0, \infty) = \left\{ u(\cdot) \in \mathcal{U}_{loc}[0, \infty) \mid \mathbb{E} \int_0^\infty |u(s)|^2 ds < \infty \right\}. \end{cases}$$

Any $u(\cdot) \in \mathcal{U}_{loc}[0, \infty)$ is called a *control process* and any $u(\cdot) \in \mathcal{U}[0, \infty)$ is called a *feasible control process*. Likewise, we define

$$\begin{cases} \mathcal{X}[0, T] = \left\{ X : [0, \infty) \times \Omega \rightarrow \mathbb{R}^n \mid \begin{array}{l} X(\cdot) \text{ is } \mathbb{F}\text{-adapted, } t \mapsto X(t, \omega) \text{ is continuous,} \\ \mathbb{E} \left[\sup_{t \in [0, T]} |X(t)|^2 \right] < \infty \end{array} \right\}, & T > 0, \\ \mathcal{X}_{loc}[0, \infty) = \bigcup_{T>0} \mathcal{X}[0, T], \\ \mathcal{X}[0, \infty) = \left\{ X(\cdot) \in \mathcal{X}_{loc}[0, \infty) \mid \mathbb{E} \int_0^\infty |X(t)|^2 dt < \infty \right\}. \end{cases}$$

Any element in $\mathcal{X}_{loc}[0, \infty)$ is called a *state process*. It is not hard to see that

$$\begin{cases} \mathcal{U}[0, \infty) \subseteq \mathcal{U}_{loc}[0, \infty), & \mathcal{U}[0, \infty) \neq \mathcal{U}_{loc}[0, \infty), \\ \mathcal{X}[0, \infty) \subseteq \mathcal{X}_{loc}[0, \infty), & \mathcal{X}[0, \infty) \neq \mathcal{X}_{loc}[0, \infty). \end{cases}$$

By a standard argument using contraction mapping theorem, one can show that for any $(x, u(\cdot)) \in \mathbb{R}^n \times \mathcal{U}_{loc}[0, \infty)$, (1.1) admits a unique solution $X(\cdot) = X(\cdot; x, u(\cdot)) \in \mathcal{X}_{loc}[0, \infty)$. Next, we let $Q, \bar{Q} \in \mathcal{S}^n$ and $R, \bar{R} \in \mathcal{S}^m$, where \mathcal{S}^k is the set of all symmetric matrices of order $(k \times k)$, and introduce the following cost functional:

$$\begin{aligned} J(x; u(\cdot)) = \mathbb{E} \int_0^\infty \Big\{ & \langle QX(s), X(s) \rangle + \langle \bar{Q}\mathbb{E}[X(s)], \mathbb{E}[X(s)] \rangle \\ & + \langle Ru(s), u(s) \rangle + \langle \bar{R}\mathbb{E}[u(s)], \mathbb{E}[u(s)] \rangle \Big\} ds, \end{aligned} \quad (1.2)$$

where $X(\cdot) = X(\cdot; x, u(\cdot))$ on the right hand side of the above. Note that in general, for $(x, u(\cdot)) \in \mathbb{R}^n \times \mathcal{U}[0, \infty)$, the solution $X(\cdot) \equiv X(\cdot; x, u(\cdot))$ of (1.1) might just be in $\mathcal{X}_{loc}[0, \infty)$ and the above cost functional $J(x; u(\cdot))$ might not be defined. Therefore, we introduce the following:

$$\mathcal{U}_{ad}[0, \infty) = \left\{ u(\cdot) \in \mathcal{U}[0, \infty) \mid J(x; u(\cdot)) \text{ is defined, } \forall x \in \mathbb{R}^n \right\}.$$

Any element $u(\cdot) \in \mathcal{U}_{ad}[0, \infty)$ is called an *admissible control process* and the corresponding $X(\cdot) \equiv X(\cdot; x, u(\cdot))$ is called an *admissible state process*. We see that the structure of $\mathcal{U}_{ad}[0, \infty)$ is very complicated, since it involves not only the state equation, but also the cost functional. Some better description of $\mathcal{U}_{ad}[0, \infty)$ will be given a little later, under proper conditions. Our optimal control problem can be stated as follows:

Problem (MF-LQ). For given $x \in \mathbb{R}^n$, find a $u_*(\cdot) \in \mathcal{U}_{ad}[0, \infty)$ such that

$$J(x; u_*(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}_{ad}[0, \infty)} J(x; u(\cdot)) \equiv V(x).$$

Any $u_*(\cdot) \in \mathcal{U}_{ad}[0, \infty)$ satisfying the above is called an *optimal control process* and the corresponding state process $X_*(\cdot) \equiv X(\cdot; x, u_*(\cdot))$ is called an *optimal state process*; the pair $(X_*(\cdot), u_*(\cdot))$ is called an *optimal pair*. The function $V(\cdot)$ is called the *value function* of Problem (MF-LQ).

It is not hard to see that in order Problem (MF-LQ) to make sense, we need $\mathcal{U}_{ad}[0, \infty)$ to be nonempty, at least. To achieve this, we will carefully discuss various stabilizability (for which both the state equation and the cost functional are involved) of the controlled MF-FSDE (1.1), which are interestingly different from the classic ones, due to the appearance of the terms $\mathbb{E}[X(\cdot)]$ and $\mathbb{E}[u(\cdot)]$. Once the set $\mathcal{U}_{ad}[0, \infty)$ of admissible controls is nonempty, under some standard assumptions, we are able to show that the optimal control uniquely exists. Then inspired by the results of [35], we obtain a system of algebraic Riccati equations (AREs, for short), whose solutions will lead us to the state feedback representation of the optimal control. The existence of the solutions to the derived ARE system is established under some reasonable conditions. Our results recovers relevant ones for the classic linear-quadratic optimal controls of SDEs.

The rest of the paper is organized as follows. Section 2 collects some preliminary results concerning the state equation. In Sections 3 and 4, the stability and the stabilizability of the state equation are discussed. In Section 5, Problem (MF-LQ) is solved by means of AREs. In Section 6, the solvability of AREs is discussed by linear matrix inequalities (LMIs, for short). A couple of numerical examples are presented in Section 7. Finally, some supporting results for Section 6 are listed in the Appendix.

2 Preliminary Results

In this section, we present some preliminary results. First of all, let us consider the following result, whose proof follows a standard argument using contraction mapping theorem, together with Itô's formula.

Proposition 2.1 *For any $x \in \mathbb{R}^n$ and $u(\cdot) \in \mathcal{U}[0, \infty)$, there exists a unique $X(\cdot) \equiv X(\cdot; x, u(\cdot))$ solving (1.1). Moreover,*

$$\mathbb{E}\left[\sup_{t \in [0, T]} |X(t)|^2\right] \leq L_T \left\{ |x|^2 + \mathbb{E} \int_0^T |u(t)|^2 dt \right\}, \quad \forall T > 0.$$

where $L_T > 0$ is a constant depending on T , and independent of $(x, u(\cdot))$.

For later purposes, we make some calculations. Let $X(\cdot) = X(\cdot; x, u(\cdot))$ be the solution of (1.1). For any deterministic differentiable function $P(\cdot)$ valued in \mathcal{S}^n , by Itô's formula, we have

$$\begin{aligned} & d \langle P(t)X(t), X(t) \rangle \\ &= \left\{ \langle \dot{P}(t)X(t), X(t) \rangle + 2 \langle P(t) \{ AX(t) + \bar{A}\mathbb{E}[X(t)] + Bu(t) + \bar{B}\mathbb{E}[u(t)] \}, X(t) \rangle \right. \\ & \quad \left. + \langle P(t) \{ CX(t) + \bar{C}\mathbb{E}[X(t)] + Du(t) + \bar{D}\mathbb{E}[u(t)] \}, CX(t) + \bar{C}\mathbb{E}[X(t)] + Du(t) + \bar{D}\mathbb{E}[u(t)] \rangle \right\} dt + \{ \cdots \} dW(t) \\ &= \left\{ \langle \dot{P}(t)X(t), X(t) \rangle + 2 \langle P(t) \{ AX(t) + Bu(t) \}, X(t) \rangle + \langle P(t) \{ CX(t) + Du(t) \}, CX(t) + Du(t) \rangle \right. \\ & \quad + 2 \langle P(t) \{ \bar{A}\mathbb{E}[X(t)] + \bar{B}\mathbb{E}[u(t)] \}, X(t) \rangle + 2 \langle P(t) \{ CX(t) + Du(t) \}, \bar{C}\mathbb{E}[X(t)] + \bar{D}\mathbb{E}[u(t)] \rangle \\ & \quad \left. + \langle P(t) \{ \bar{C}\mathbb{E}[X(t)] + \bar{D}\mathbb{E}[u(t)] \}, \bar{C}\mathbb{E}[X(t)] + \bar{D}\mathbb{E}[u(t)] \rangle \right\} dt + \{ \cdots \} dW(t). \end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{E} \langle P(t)X(t), X(t) \rangle &= \langle P(0)x, x \rangle + \mathbb{E} \int_0^t \left\{ \langle [\dot{P}(s) + P(s)A + A^T P(s) + C^T P(s)C] X(s), X(s) \rangle \right. \\
&\quad + 2 \langle u(s), (B^T P(s) + D^T P(s)C) X(s) \rangle + \langle D^T P(s)Du(s), u(s) \rangle \\
&\quad + \langle [P(s)\bar{A} + \bar{A}^T P(s) + \bar{C}^T P(s)\bar{C} + \bar{C}^T P(s)C + C^T P(s)\bar{C}] \mathbb{E}[X(s)], \mathbb{E}[X(s)] \rangle \\
&\quad + 2 \langle \mathbb{E}[u(s)], [\bar{B}^T P(s) + \bar{D}^T P(s)\bar{C} + \bar{D}^T P(s)C + D^T P(s)\bar{C}] \mathbb{E}[X(s)] \rangle \\
&\quad \left. + \langle [\bar{D}^T P(s)\bar{D} + \bar{D}^T P(s)D + D^T P(s)\bar{D}] \mathbb{E}[u(s)], \mathbb{E}[u(s)] \rangle \right\} ds.
\end{aligned} \tag{2.1}$$

Also,

$$\begin{aligned}
\langle P(t)\mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle &= \langle P(0)x, x \rangle + \int_0^t \left\{ \langle [\dot{P}(s) + P(s)(A + \bar{A}) + (A + \bar{A})^T P(s)] \mathbb{E}[X(s)], \mathbb{E}[X(s)] \rangle \right. \\
&\quad \left. + 2 \langle \mathbb{E}[u(s)], (B + \bar{B})^T P(s) \mathbb{E}[X(s)] \rangle \right\} ds.
\end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2), we obtain

$$\begin{aligned}
\mathbb{E} \langle P(t) \{ X(t) - \mathbb{E}[X(t)] \}, X(t) - \mathbb{E}[X(t)] \rangle &= \mathbb{E} \langle P(t)X(t), X(t) \rangle - \langle P(t)\mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle \\
&= \mathbb{E} \int_0^t \left\{ \langle [\dot{P}(s) + P(s)A + A^T P(s) + C^T P(s)C] X(s), X(s) \rangle + 2 \langle u(s), [B^T P(s) + D^T P(s)C] X(s) \rangle \right. \\
&\quad + \langle D^T P(s)Du(s), u(s) \rangle + \langle [P(s)\bar{A} + \bar{A}^T P(s) + \bar{C}^T P(s)\bar{C} + \bar{C}^T P(s)C + C^T P(s)\bar{C}] \mathbb{E}[X(s)], \mathbb{E}[X(s)] \rangle \\
&\quad + 2 \langle \mathbb{E}[u(s)], [\bar{B}^T P(s) + \bar{D}^T P(s)\bar{C} + \bar{D}^T P(s)C + D^T P(s)\bar{C}] \mathbb{E}[X(s)] \rangle \\
&\quad + \langle [\bar{D}^T P(s)\bar{D} + \bar{D}^T P(s)D + D^T P(s)\bar{D}] \mathbb{E}[u(s)], \mathbb{E}[u(s)] \rangle \\
&\quad \left. - \langle [\dot{P}(s) + P(s)(A + \bar{A}) + (A + \bar{A})^T P(s)] \mathbb{E}[X(s)], \mathbb{E}[X(s)] \rangle - 2 \langle \mathbb{E}[u(s)], (B + \bar{B})^T P(s) \mathbb{E}[X(s)] \rangle \right\} ds \\
&= \mathbb{E} \int_0^t \left\{ \langle [\dot{P}(s) + P(s)A + A^T P(s) + C^T P(s)C] \{ X(s) - \mathbb{E}[X(s)] \}, X(s) - \mathbb{E}[X(s)] \rangle \right. \\
&\quad + 2 \langle u(s) - \mathbb{E}[u(s)], [B^T P(s) + D^T P(s)C] \{ X(s) - \mathbb{E}[X(s)] \} \rangle \\
&\quad + \langle D^T P(s)D \{ u(s) - \mathbb{E}[u(s)] \}, u(s) - \mathbb{E}[u(s)] \rangle \\
&\quad \left. + \langle P(s) \{ (C + \bar{C}) \mathbb{E}[X(s)] + (D + \bar{D}) \mathbb{E}[u(s)] \}, (C + \bar{C}) \mathbb{E}[X(s)] + (D + \bar{D}) \mathbb{E}[u(s)] \rangle \right\} ds.
\end{aligned}$$

In the case that $P(t) \equiv P \in \mathcal{S}^n$, we have

$$\begin{aligned}
\mathbb{E} \langle PX(t), X(t) \rangle &= \langle Px, x \rangle + \mathbb{E} \int_0^t \left\{ \langle (PA + A^T P + C^T PC) X(s), X(s) \rangle \right. \\
&\quad + 2 \langle u(s), (B^T P + D^T PC) X(s) \rangle + \langle D^T P Du(s), u(s) \rangle \\
&\quad + \langle (P\bar{A} + \bar{A}^T P + \bar{C}^T P\bar{C} + \bar{C}^T PC + C^T P\bar{C}) \mathbb{E}[X(s)], \mathbb{E}[X(s)] \rangle \\
&\quad + 2 \langle \mathbb{E}[u(s)], (\bar{B}^T P + \bar{D}^T P\bar{C} + \bar{D}^T PC + D^T P\bar{C}) \mathbb{E}[X(s)] \rangle \\
&\quad \left. + \langle (\bar{D}^T P\bar{D} + \bar{D}^T PD + D^T P\bar{D}) \mathbb{E}[u(s)], \mathbb{E}[u(s)] \rangle \right\} ds,
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
\langle P\mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle &= \langle Px, x \rangle + \int_0^t \left\{ \langle [P(A + \bar{A}) + (A + \bar{A})^T P] \mathbb{E}[X(s)], \mathbb{E}[X(s)] \rangle \right. \\
&\quad \left. + 2 \langle \mathbb{E}[u(s)], (B + \bar{B})^T P \mathbb{E}[X(s)] \rangle \right\} ds
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E} \langle P \{X(t) - \mathbb{E}[X(t)]\}, X(t) - \mathbb{E}[X(t)] \rangle \\
&= \mathbb{E} \int_0^t \left\{ \langle (PA + A^T P + C^T PC) \{X(s) - \mathbb{E}[X(s)]\}, X(s) - \mathbb{E}[X(s)] \rangle \right. \\
&\quad + 2 \langle u(s) - \mathbb{E}[u(s)], (B^T P + D^T PC) \{X(s) - \mathbb{E}[X(s)]\} \rangle + \langle D^T P D \{u(s) - \mathbb{E}[u(s)]\}, u(s) - \mathbb{E}[u(s)] \rangle \\
&\quad \left. + \langle P \{ (C + \bar{C}) \mathbb{E}[X(s)] + (D + \bar{D}) \mathbb{E}[u(s)] \}, (C + \bar{C}) \mathbb{E}[X(s)] + (D + \bar{D}) \mathbb{E}[u(s)] \rangle \right\} ds.
\end{aligned} \tag{2.4}$$

The above will be useful later.

Now, let us look at the cost functional. We observe that the cost functional $J(x; u(\cdot))$ defined by (1.2) can also be written as

$$\begin{aligned}
J(x; u(\cdot)) &= \mathbb{E} \int_0^\infty \left\{ \langle Q \{X(t) - \mathbb{E}[X(t)]\}, X(t) - \mathbb{E}[X(t)] \rangle + \langle (Q + \bar{Q}) \mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle \right. \\
&\quad \left. + \langle R \{u(t) - \mathbb{E}[u(t)]\}, u(t) - \mathbb{E}[u(t)] \rangle + \langle (R + \bar{R}) \mathbb{E}[u(t)], \mathbb{E}[u(t)] \rangle \right\} dt.
\end{aligned}$$

In what follows, when the dimension of a matrix, say, Q is clear from the context, we write $Q \geq 0$ for $Q \in \mathcal{S}^n$ being positive semi-definite and write $Q > 0$ for $Q \in \mathcal{S}^n$ being positive definite. We now introduce the following assumption concerning the weighting matrices Q, \bar{Q}, R, \bar{R} in the cost functional.

(J) The matrices $Q, \bar{Q} \in \mathcal{S}^n$ and $R, \bar{R} \in \mathcal{S}^m$ satisfy the following:

$$Q, Q + \bar{Q} \geq 0, \quad R, R + \bar{R} > 0.$$

Note that in (J), we do not have direct assumption on \bar{Q} and \bar{R} , they do not have to be positive (semi-) definite, and actually, they could even be negative definite. Under (J), we see that $u(\cdot) \in \mathcal{U}_{ad}[0, \infty)$ if and only if for any $x \in \mathbb{R}^n$, the corresponding state process $X(\cdot) \equiv X(\cdot; x, u(\cdot))$ satisfies

$$\mathbb{E} \int_0^\infty \left(|Q^{\frac{1}{2}} \{X(t) - \mathbb{E}[X(t)]\}|^2 + |(Q + \bar{Q})^{\frac{1}{2}} \mathbb{E}[X(t)]|^2 \right) dt < \infty. \tag{2.5}$$

Since Q and/or $(Q + \bar{Q})$ might be degenerate, when $u(\cdot) \in \mathcal{U}_{ad}[0, \infty)$, we might not have $X(\cdot) \equiv X(\cdot; x, u(\cdot)) \in \mathcal{X}[0, \infty)$. The following is a little stronger assumption than (J).

(J)' The matrices $Q, \bar{Q} \in \mathcal{S}^n$ and $R, \bar{R} \in \mathcal{S}^m$ satisfy the following:

$$Q, Q + \bar{Q} > 0, \quad R, R + \bar{R} > 0.$$

Clearly, if (J)' holds, then $u(\cdot) \in \mathcal{U}_{ad}[0, \infty)$ if and only if for any $x \in \mathbb{R}^n$, $X(\cdot; x, u(\cdot)) \in \mathcal{X}[0, \infty)$.

3 Stability

Now, let us return to state equation (1.1). We know that cost functional $J(x; u(\cdot))$ is well-defined on $\mathbb{R}^n \times \mathcal{U}_{ad}[0, \infty)$, and unlike $\mathcal{U}[0, \infty)$, the structure of $\mathcal{U}_{ad}[0, \infty)$ seems to be complicated since it involves the state equation and the cost functional. Further, the following example shows that $\mathcal{U}_{ad}[0, \infty)$ could even be empty, which leads to that Problem (MF-LQ) is meaningless.

Example 3.1 Consider one-dimensional controlled system:

$$dX(t) = X(t)dt + \{\mathbb{E}[X(t)] + u(t)\}dW(t), \quad t \geq 0,$$

with cost functional

$$J(x; u(\cdot)) = \mathbb{E} \int_0^\infty |X(t)|^2 dt.$$

Clearly,

$$d\mathbb{E}[X(t)] = \mathbb{E}[X(t)]dt, \quad t \geq 0,$$

which implies

$$\mathbb{E}[X(t)] = xe^t, \quad t \geq 0.$$

Then

$$dX(t) = X(t)dt + [xe^t + u(t)]dW(t), \quad t \geq 0.$$

Hence,

$$X(t) = xe^t + \int_0^t e^{t-s} [xe^s + u(s)]dW(s) = e^t \left\{ x + \int_0^t [x + e^{-s}u(s)]dW(s) \right\}, \quad t \geq 0,$$

and as long as $x \neq 0$ or $u(\cdot) \neq 0$,

$$J(x; u(\cdot)) = \mathbb{E} \int_0^\infty |X(t)|^2 dt = \int_0^\infty e^{2t} \left\{ x^2 + \int_0^t [x + e^{-s}u(s)]^2 ds \right\} dt = \infty.$$

Therefore, in this case, $\mathcal{U}_{ad}[0, \infty) = \emptyset$. Consequently, the corresponding Problem (MF-LQ) is not meaningful.

From the above, we see that before investigating Problem (MF-LQ), we should find conditions for the system and the cost functional so that the set $\mathcal{U}_{ad}[0, \infty)$ is at least non-empty and hopefully it admits an accessible characterization. To this end, let us first look at the following uncontrolled linear MF-FSDE (which amount to saying that taking $u(\cdot) = 0$ or letting $B = \bar{B} = D = \bar{D} = 0$):

$$\begin{cases} dX(t) = \{AX(t) + \bar{A}\mathbb{E}[X(t)]\}dt + \{CX(t) + \bar{C}\mathbb{E}[X(t)]\}dW(t), & t \geq 0, \\ X(0) = x, \end{cases} \quad (3.1)$$

where $A, \bar{A}, C, \bar{C} \in \mathbb{R}^{n \times n}$ are given matrices. The above uncontrolled system is briefly denoted by $[A, \bar{A}, C, \bar{C}]$. For simplicity, we also denote $[A, C] = [A, 0, C, 0]$ (the linear SDE without mean-fields), and $A = [A, 0] \equiv [A, 0, 0, 0]$ (the linear ordinary differential equation, ODE, for short). Let us now introduce the following definition.

Definition 3.2 (i) System $[A, \bar{A}, C, \bar{C}]$ is said to be L^2 -*exponentially stable* if for any $x \in \mathbb{R}^n$, the solution $X(\cdot) \equiv X(\cdot; x) \in \mathcal{X}_{loc}[0, \infty)$ of (3.1) satisfies the following:

$$\lim_{t \rightarrow \infty} e^{\lambda t} \mathbb{E}|X(t)|^2 = 0,$$

for some $\lambda > 0$.

(ii) System $[A, \bar{A}, C, \bar{C}]$ is said to be L^2 -*globally integrable* if for any $x \in \mathbb{R}^n$, the solution $X(\cdot) \equiv X(\cdot; x) \in \mathcal{X}_{loc}[0, \infty)$ of (3.1) is in $\mathcal{X}[0, \infty)$, namely,

$$\int_0^\infty \mathbb{E}|X(t)|^2 dt < \infty.$$

(iii) System $[A, \bar{A}, C, \bar{C}]$ is said to be L^2 -*asymptotically stable* if for any $x \in \mathbb{R}^n$, the solution $X(\cdot) \equiv X(\cdot; x) \in \mathcal{X}_{loc}[0, \infty)$ of (3.1) satisfies the following:

$$\lim_{t \rightarrow \infty} \mathbb{E}|X(t)|^2 = 0. \quad (3.2)$$

(iv) Let (J) hold. System $[A, \bar{A}, C, \bar{C}]$ is said to be $L^2_{Q, \bar{Q}}$ -*globally integrable* if for any $x \in \mathbb{R}^n$, the solution $X(\cdot) \equiv X(\cdot; x) \in \mathcal{X}_{loc}[0, \infty)$ of (3.1) satisfies (2.5).

It is clear that the above (i)–(iii) can be defined for linear SDE system $[A, C] = [A, 0, C, 0]$, and linear ODE system $A = [A, 0]$. By a standard result, we know that the above (i)–(iii) are equivalent for linear ODEs. For general linear MF-SDEs, we have the following result.

Proposition 3.3 *Among the following statements:*

- (i) System $[A, \bar{A}, C, \bar{C}]$ is L^2 -exponentially stable;
- (ii) System $[A, \bar{A}, C, \bar{C}]$ is L^2 -globally integrable;
- (iii) System $[A, \bar{A}, C, \bar{C}]$ is L^2 -asymptotically stable;
- (iv) Let (J) hold. System $[A, \bar{A}, C, \bar{C}]$ is $L^2_{Q, \bar{Q}}$ -globally integrable.

The following implications hold:

$$\begin{aligned} (i) &\Rightarrow (ii) \Rightarrow (iii); \\ \text{when (J) holds, } (ii) &\Rightarrow (iv); \quad \text{when (J)' holds, } (iv) \Rightarrow (ii). \end{aligned}$$

Proof. The implications (i) \Rightarrow (ii) and (ii) \Rightarrow (iv) (under (J)) are clear. It is also clear that under (J)', (iv) \Rightarrow (ii). We now prove (ii) \Rightarrow (iii). Note that

$$\begin{aligned} \mathbb{E}|X(t)|^2 &= |x|^2 + \mathbb{E} \int_0^t \left(2 \langle X(s), AX(s) + \bar{A}\mathbb{E}[X(s)] \rangle + |CX(s) + \bar{C}\mathbb{E}[X(s)]|^2 \right) ds \\ &\leq |x|^2 + L \int_0^t \mathbb{E}|X(s)|^2 ds \leq |x|^2 + L \mathbb{E} \int_0^\infty |X(s)|^2 ds. \end{aligned}$$

Hereafter $L > 0$ stands for a generic constant which could be different from line to line. Thus, $\mathbb{E}|X(t)|^2$ is bounded uniformly in $t \in [0, \infty)$. Consequently, for any $0 \leq \tau < t < \infty$,

$$\begin{aligned} \left| \mathbb{E}|X(t)|^2 - \mathbb{E}|X(\tau)|^2 \right| &\leq \mathbb{E} \int_\tau^t \left(2 \left| \langle X(s), AX(s) + \bar{A}\mathbb{E}[X(s)] \rangle \right| + |CX(s) + \bar{C}\mathbb{E}[X(s)]|^2 \right) ds \\ &\leq L(t - \tau). \end{aligned}$$

Hence, $t \mapsto \mathbb{E}|X(t)|^2$ is uniformly continuous on $[0, \infty)$, which, together with the integrability of $\mathbb{E}|X(\cdot)|^2$ over $[0, \infty)$, leads to (3.2). \square

Let us make the following remarks.

- When (J) holds but (J)' does not hold, the $L^2_{Q, \bar{Q}}$ -global integrability of the system does not imply the L^2 -global integrability of the system in general.
- It is not clear if (iii) implies (ii), although these two are equivalent for ODE case.
- The notion that is the most relevant to our Problem (MF-LQ) is the $L^2_{Q, \bar{Q}}$ -global integrability.

Our next goal is to explore when (ii) implies (i). To this end, we first look the case $\bar{A} = \bar{C} = 0$. In this case, our system becomes system $[A, C]$:

$$\begin{cases} dX(t) = AX(t)dt + CX(t)dW(t), & t \geq 0, \\ X(0) = x. \end{cases} \quad (3.3)$$

For such a system, instead of $L^2_{Q, \bar{Q}}$ -global integrability, we may introduce the following.

Definition 3.4 Let $Q \geq 0$. System $[A, C]$ is said to be L_Q^2 -globally integrable if for any $x \in \mathbb{R}^n$, the solution $X(\cdot) \equiv X(\cdot; x)$ of (3.3) satisfies

$$\mathbb{E} \int_0^\infty \langle QX(t), X(t) \rangle dt < \infty.$$

In the case that $Q > 0$, the L_Q^2 -global integrability is simply called the L^2 -global integrability which is equivalent to $X(\cdot; x) \in \mathcal{X}[0, \infty)$ for all $x \in \mathbb{R}^n$.

We have the following result concerning the L_Q^2 -global integrability of $[A, C]$.

Proposition 3.5 *Let $Q \geq 0$. Then the following are equivalent:*

- (i) *System $[A, C]$ is L_Q^2 -globally integrable;*
- (ii) *The following Lyapunov equation admits a solution $P \geq 0$:*

$$PA + A^T P + C^T P C + Q = 0; \quad (3.4)$$

- (iii) *The Lyapunov equation (3.4) admits a solution $P \in \mathcal{S}^n$ and*

$$\overline{\lim}_{t \rightarrow \infty} \mathbb{E}|X(t; x)|^2 < \infty, \quad \forall x \in \mathbb{R}^n,$$

which is the case, in particular, if $[A, C]$ is L^2 -asymptotically stable.

In the above case, the solution P of the above equation admits the following representation:

$$P = \mathbb{E} \int_0^\infty \bar{F}(t)^T Q \bar{F}(t) dt, \quad (3.5)$$

where $\bar{F}(\cdot)$ is the solution to the following:

$$\begin{cases} d\bar{F}(t) = A\bar{F}(t)dt + C\bar{F}(t)dW(t), & t \in [0, \infty), \\ \bar{F}(0) = I. \end{cases}$$

The above result should be standard. However, since the idea contained in the proof will be useful below, for readers's convenience, we present a proof here.

Proof. (i) \Rightarrow (ii). Suppose system $[A, C]$ is L_Q^2 -globally integrable. We want to show that Lyapunov equation (3.4) admits a solution $P \geq 0$. To this end, let us consider the following linear ODE:

$$\begin{cases} -\dot{\Theta}(t) + \Theta(t)A + A^T \Theta(t) + C^T \Theta(t)C + Q = 0, & t \in [0, \infty), \\ \Theta(0) = 0, \end{cases} \quad (3.6)$$

which has a unique solution $\Theta(\cdot)$ defined on $[0, \infty)$. For any fixed $\tau > 0$, we define

$$\bar{\Theta}^\tau(s) = \Theta(\tau - s), \quad s \in [0, \tau].$$

Then $\bar{\Theta}^\tau(\cdot)$ is the solution to the following:

$$\begin{cases} \dot{\bar{\Theta}}^\tau(s) + \bar{\Theta}^\tau(s)A + A^T \bar{\Theta}^\tau(s) + C^T \bar{\Theta}^\tau(s)C + Q = 0, & s \in [0, \tau], \\ \bar{\Theta}^\tau(\tau) = 0. \end{cases}$$

For any $x \in \mathbb{R}^n$, let $X(\cdot) \equiv X(\cdot; x)$ be the solution of (3.3). Applying Itô's formula to $s \mapsto \langle \bar{\Theta}^\tau(s)X(s), X(s) \rangle$, one has

$$\begin{aligned} -\langle \Theta(\tau)x, x \rangle &= -\langle \bar{\Theta}^\tau(0)x, x \rangle = \mathbb{E} \left[\langle \bar{\Theta}^\tau(\tau)X(\tau), X(\tau) \rangle - \langle \bar{\Theta}^\tau(0)X(0), X(0) \rangle \right] \\ &= \mathbb{E} \int_0^\tau \langle \{ \dot{\bar{\Theta}}^\tau(s) + \bar{\Theta}^\tau(s)A + A^T \bar{\Theta}^\tau(s) + C^T \bar{\Theta}^\tau(s)C \} X(s), X(s) \rangle ds \\ &= -\mathbb{E} \int_0^\tau \langle QX(s), X(s) \rangle ds = -\mathbb{E} \int_0^\tau \langle \bar{F}(s)^T Q \bar{F}(s)x, x \rangle ds. \end{aligned}$$

Thus, the solution $\Theta(\cdot)$ of (3.6) admits the following representation:

$$\Theta(\tau) = \mathbb{E} \int_0^\tau \bar{F}(s)^T Q \bar{F}(s) ds, \quad \tau \geq 0.$$

From the above, since $Q \geq 0$, we see that $\tau \mapsto \Theta(\tau)$ is non-decreasing and by the L_Q^2 -global integrability of $[A, C]$, one has the following limit:

$$\lim_{\tau \rightarrow \infty} \Theta(\tau) = \mathbb{E} \int_0^\infty \bar{F}(s)^T Q \bar{F}(s) ds \equiv P.$$

We claim that such a $P \geq 0$ must be a solution to the Lyapunov equation (3.4). In fact, from (3.6), one has

$$\Theta(t) - \Theta(t+1) + \left(\int_t^{t+1} \Theta(s) ds \right) A + A^T \left(\int_t^{t+1} \Theta(s) ds \right) + C^T \left(\int_t^{t+1} \Theta(s) ds \right) C + Q = 0.$$

Letting $t \rightarrow \infty$, we see that (3.4) is satisfied by P .

(ii) \Rightarrow (i) Suppose there exists a $P \geq 0$ satisfying (3.4). Then

$$\begin{aligned} \mathbb{E} \langle PX(t), X(t) \rangle - \langle Px, x \rangle &= \mathbb{E} \int_0^t \langle (PA + A^T P + C^T P C) X(s), X(s) \rangle ds \\ &= -\mathbb{E} \int_0^t \langle QX(s), X(s) \rangle ds. \end{aligned} \tag{3.7}$$

This implies

$$\mathbb{E} \int_0^t \langle QX(s), X(s) \rangle ds = \langle Px, x \rangle - \mathbb{E} \langle PX(t), X(t) \rangle \leq \langle Px, x \rangle, \quad t \geq 0. \tag{3.8}$$

Thus, the system is L_Q^2 -globally integrable.

(i) \Rightarrow (iii) is clear.

(iii) \Rightarrow (i). Suppose (3.4) has a solution $P \in \mathcal{S}^n$. Then by (3.8), we have

$$\mathbb{E} \int_0^t \langle QX(s), X(s) \rangle ds \leq \langle Px, x \rangle + L \mathbb{E} |X(t)|^2.$$

Hence, $[A, C]$ is L_Q^2 -globally integrable. □

Combining Propositions 3.3 and 3.5, we have the following result for system $[A, C]$.

Proposition 3.6 *The following are equivalent:*

- (i) System $[A, C]$ is L^2 -exponentially stable;
- (ii) System $[A, C]$ is L^2 -globally integrable;
- (iii) For any $Q > 0$, the Lyapunov equation (3.4) admits a solution $P > 0$, and in this case, the representation (3.5) holds for this P ;
- (iv) System $[A, C]$ is L^2 -asymptotically stable, and for some $Q > 0$, Lyapunov equation (3.4) admits a solution $P \in \mathcal{S}^n$.

Proof. The implications (i) \Rightarrow (ii) \Rightarrow is clear. The relations (ii) \Longleftrightarrow (iii) \Longleftrightarrow (iv) follow from Proposition 3.5. The implication (iii) \Rightarrow (i) follows from (3.7), together with the positive definiteness of P and Q and Gronwall's inequality. \square

Now, let us return system $[A, \bar{A}, C, \bar{C}]$. We have the following result.

Theorem 3.7 (i) Suppose system $[A, \bar{A}, C, \bar{C}]$ is L^2 -asymptotically stable. Then it is necessary that $A + \bar{A}$ is exponentially stable.

(ii) If $A + \bar{A}$ is exponentially stable, then system $[A, \bar{A}, C, \bar{C}]$ is L^2 -exponentially stable if either $[A, C]$ is L^2 -globally integrable, or

$$C + \bar{C} = 0. \quad (3.9)$$

Proof. (i) Suppose (3.2) holds. Taking expectation in (3.1), we obtain

$$\begin{cases} d\mathbb{E}[X(t)] = (A + \bar{A})\mathbb{E}[X(t)]dt, & t \geq 0, \\ \mathbb{E}[X(0)] = x. \end{cases} \quad (3.10)$$

Hence,

$$\mathbb{E}[X(t)] = e^{(A+\bar{A})t}x, \quad t \geq 0.$$

Since

$$|\mathbb{E}[X(t)]|^2 \leq \mathbb{E}[X(t)]^2, \quad t \geq 0,$$

the L^2 -asymptotic stability of system $[A, \bar{A}, C, \bar{C}]$ implies the exponential stability of $A + \bar{A}$.

(ii) By (2.4) with $B = \bar{B} = D = \bar{D} = 0$, we have, for any $P \in \mathcal{S}^n$,

$$\begin{aligned} & \mathbb{E} \langle P \{X(t) - \mathbb{E}[X(t)]\}, X(t) - \mathbb{E}[X(t)] \rangle \\ &= \mathbb{E} \int_0^t \left\{ \langle (PA + A^T P + C^T P C) \{X(s) - \mathbb{E}[X(s)]\}, X(s) - \mathbb{E}[X(s)] \rangle \right. \\ & \quad \left. + \langle P(C + \bar{C})\mathbb{E}[X(s)], (C + \bar{C})\mathbb{E}[X(s)] \rangle \right\} ds. \end{aligned} \quad (3.11)$$

Hence, if (3.9) holds, one has from the above that

$$\text{var}[X(t)] \leq L \int_0^t \text{var}[X(s)]ds, \quad \forall t \geq 0.$$

Then, by Gronwall's inequality, we obtain

$$\text{var}[X(t)] = 0, \quad t \geq 0.$$

Consequently, if we let $2\lambda = -\max \sigma(A + \bar{A}) > 0$, then

$$e^{2\lambda t} \mathbb{E}|X(t)|^2 = e^{2\lambda t} \left(\text{var}[X(t)] + |\mathbb{E}[X(t)]|^2 \right) = |e^{\lambda t} e^{(A+\bar{A})t}x|^2 \rightarrow 0, \quad t \rightarrow \infty.$$

Thus, $[A, \bar{A}, C, \bar{C}]$ is L^2 -exponentially stable.

Next, if $[A, C]$ is L^2 -globally integrable, then by Proposition 3.6, for $Q = I$, there exists a $P > 0$ such that

$$PA + A^T P + C^T P C + I = 0.$$

Hence, (3.11) implies

$$\text{var}[X(t)] \leq -\mu \int_0^t \text{var}[X(s)]ds + L \int_0^t |\mathbb{E}[X(s)]|^2 ds, \quad t \geq 0,$$

for some $\mu, L > 0$, with $\mu \neq \lambda = -\max \sigma(A + \bar{A}) > 0$. By Gronwall's inequality,

$$\text{var}[X(t)] \leq L|x|^2 \int_0^t e^{-\mu(t-s)} e^{-\lambda s} ds = L|x|^2 \frac{e^{-\lambda t} - e^{-\mu t}}{\mu - \lambda}, \quad t \geq 0.$$

This results in

$$\mathbb{E}|X(t)|^2 = \text{var}[X(t)] + |\mathbb{E}[X(t)]|^2 \leq L|x|^2 \frac{e^{-\lambda t} - e^{-\mu t}}{\mu - \lambda} + |e^{(A+\bar{A})t}x|^2, \quad t \geq 0.$$

Therefore, the system $[A, \bar{A}, C, \bar{C}]$ is L^2 -exponentially stable. This completes the proof. \square

Note that the exponential stability of $A + \bar{A}$ together with the L^2 -global integrability of $[A, C]$ or (3.9) are sufficient conditions for the L^2 -exponential stability of system $[A, \bar{A}, C, \bar{C}]$. When $n = 1$, these conditions are also necessary in some sense. To be more precise, let us look at the following one-dimensional system:

$$\begin{cases} dX(t) = \{aX(t) + \bar{a}\mathbb{E}[X(t)]\}dt + \{cX(t) + \bar{c}\mathbb{E}[X(t)]\}dW(t), & t \geq 0, \\ X(0) = x. \end{cases} \quad (3.12)$$

We have the following result.

Proposition 3.8 *For system (3.12), the following are equivalent:*

- (i) *It is L^2 -exponentially stable;*
- (ii) *It is L^2 -globally integrable;*
- (iii) *It is L^2 -asymptotically stable;*
- (iv) *$a + \bar{a} < 0$, and*

$$\text{either } 2a + c^2 < 0, \quad \text{or} \quad 2a + c^2 \geq 0 \text{ and } c + \bar{c} = 0.$$

Proof. It suffices to prove the implication (iii) \Rightarrow (iv). By (2.3) with $P = 1$, $B = \bar{B} = D = \bar{D} = 0$, $A = a$, $\bar{A} = \bar{a}$, $C = c$, $\bar{C} = \bar{c}$, we have

$$\begin{aligned} \mathbb{E}|X(t)|^2 &= x^2 + \mathbb{E} \int_0^t \left\{ (2a + c^2)|X(s)|^2 + (2\bar{a} + \bar{c}^2 + 2\bar{c}c)(\mathbb{E}[X(s)])^2 \right\} ds \\ &= x^2 + \int_0^t \left\{ (2a + c^2)\mathbb{E}|X(s)|^2 + [2\bar{a} - c^2 + (c + \bar{c})^2]x^2 e^{2(a+\bar{a})s} \right\} ds. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbb{E}|X(t)|^2 &= e^{(2a+c^2)t}x^2 + [2\bar{a} - c^2 + (c + \bar{c})^2]x^2 \int_0^t e^{(2a+c^2)(t-s)} e^{2(a+\bar{a})s} ds \\ &= e^{(2a+c^2)t}x^2 + (2\bar{a} - c^2)x^2 e^{(2a+c^2)t} \int_0^t e^{(2\bar{a}-c^2)s} ds + (c + \bar{c})^2 x^2 \int_0^t e^{(2a+c^2)(t-s)} e^{2(a+\bar{a})s} ds \\ &= e^{(2a+c^2)t}x^2 + x^2 e^{(2a+c^2)t} [e^{(2\bar{a}-c^2)t} - 1] + (c + \bar{c})^2 x^2 \int_0^t e^{(2a+c^2)(t-s)} e^{2(a+\bar{a})s} ds \\ &= x^2 e^{2(a+\bar{a})t} + (c + \bar{c})^2 x^2 \int_0^t e^{(2a+c^2)(t-s)} e^{2(a+\bar{a})s} ds. \end{aligned}$$

Now, if (3.2) holds, then we must have

$$a + \bar{a} < 0,$$

and

$$(c + \bar{c})^2 \int_0^t e^{(2a+c^2)(t-s)} e^{2(a+\bar{a})s} ds \rightarrow 0.$$

Thus, under $a + \bar{a} < 0$, if $c + \bar{c} \neq 0$, then we need

$$\int_0^t e^{(2a+c^2)(t-s)} e^{2(a+\bar{a})s} ds = e^{(2a+c^2)t} \int_0^t e^{(2\bar{a}-c^2)s} ds \rightarrow 0.$$

Since $\int_0^t e^{(2\bar{a}-c^2)s} ds$ is increasing, the above must lead to $2a + c^2 < 0$. Also, if $2a + c^2 \geq 0$, we must have $c + \bar{c} = 0$. This completes the proof. \square

Now, for the $L^2_{Q,\bar{Q}}$ -global integrability of system $[A, \bar{A}, C, \bar{C}]$, we have the following result.

Proposition 3.9 *Let (J) hold. If $[A, \bar{A}, C, \bar{C}]$ is $L^2_{Q,\bar{Q}}$ -globally integrable, then $A + \bar{A}$ is $L^2_{Q+\bar{Q}}$ -globally integrable, i.e.,*

$$\int_0^\infty |(Q + \bar{Q})^{\frac{1}{2}} e^{(A+\bar{A})t}|^2 dt < \infty. \quad (3.13)$$

Conversely, if (3.13) hold, then $[A, \bar{A}, C, \bar{C}]$ is $L^2_{Q,\bar{Q}}$ -globally integrable provided either (3.9) holds, or $[A, C]$ is L^2_Q -globally integrable and

$$\mathcal{N}(Q + \bar{Q}) \subseteq \mathcal{N}(C + \bar{C}), \quad (3.14)$$

where $\mathcal{N}(G)$ is the null space of G .

Proof. Since,

$$\int_0^\infty \langle (Q + \bar{Q})\mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle dt \leq \mathbb{E} \int_0^\infty \left(\langle QX(t), X(t) \rangle + \langle \bar{Q}\mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle \right) dt < \infty,$$

we see that (3.13) follows.

Next, let (3.13) hold. If (3.9) holds, we have (see (3.11) with $P = I$)

$$\begin{aligned} \text{var}[X(t)] &= \mathbb{E} \langle \{X(t) - \mathbb{E}[X(t)]\}, X(t) - \mathbb{E}[X(t)] \rangle \\ &= \mathbb{E} \int_0^t \left\{ \langle (A + A^T + C^T C) \{X(s) - \mathbb{E}[X(s)]\}, X(s) - \mathbb{E}[X(s)] \rangle + |(C + \bar{C})\mathbb{E}[X(s)]|^2 \right\} ds \\ &= \mathbb{E} \int_0^t \langle (A + A^T + C^T C) \{X(s) - \mathbb{E}[X(s)]\}, X(s) - \mathbb{E}[X(s)] \rangle ds \leq L \int_0^t \text{var}[X(s)] ds. \end{aligned}$$

Hence, by Gronwall's inequality, we obtain

$$\text{var}[X(t)] = 0, \quad t \geq 0.$$

Consequently,

$$\begin{aligned} &\mathbb{E} \int_0^\infty \left(\langle QX(t), X(t) \rangle + \langle \bar{Q}\mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle \right) dt \\ &= \mathbb{E} \int_0^\infty \left(\langle Q\{X(t) - \mathbb{E}[X(t)]\}, X(t) - \mathbb{E}[X(t)] \rangle + \langle (Q + \bar{Q})\mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle \right) dt \\ &\leq \int_0^\infty \left(|Q| \text{var}[X(t)] + |(Q + \bar{Q})^{\frac{1}{2}} e^{(A+\bar{A})t} x|^2 \right) dt \\ &= \int_0^\infty |(Q + \bar{Q})^{\frac{1}{2}} e^{(A+\bar{A})t} x|^2 dt < \infty, \end{aligned}$$

which gives the $L^2_{Q,\bar{Q}}$ -global integrability.

Finally, if (3.13) holds and $[A, C]$ is L_Q^2 -globally integrable, then by Proposition 3.5, we can find a $P \geq 0$ solving Lyapunov equation (3.4). Let $X(\cdot)$ be the solution of (3.1). Applying Itô's formula to $\langle PX(\cdot), X(\cdot) \rangle$, we get

$$\begin{aligned} & \mathbb{E} \langle P\{X(t) - \mathbb{E}[X(t)]\}, X(t) - \mathbb{E}[X(t)] \rangle \\ &= \mathbb{E} \int_0^t \left\{ \langle (PA + A^T P + C^T P C)\{X(s) - \mathbb{E}[X(s)]\}, X(s) - \mathbb{E}[X(s)] \rangle \right. \\ & \quad \left. + \langle (C + \bar{C})^T P (C + \bar{C}) \mathbb{E}[X(s)], \mathbb{E}[X(s)] \rangle \right\} ds \\ &= \mathbb{E} \int_0^t \left\{ -\langle Q\{X(s) - \mathbb{E}[X(s)]\}, X(s) - \mathbb{E}[X(s)] \rangle + \langle (C + \bar{C})^T P (C + \bar{C}) \mathbb{E}[X(s)], \mathbb{E}[X(s)] \rangle \right\} ds. \end{aligned}$$

Now, condition (3.14) implies that

$$\langle P(C + \bar{C})y, (C + \bar{C})y \rangle \leq L \langle (Q + \bar{Q})y, y \rangle, \quad \forall y \in \mathbb{R}^n,$$

for some $L > 0$. Thus,

$$\begin{aligned} & \mathbb{E} \int_0^t \langle Q\{X(s) - \mathbb{E}[X(s)]\}, X(s) - \mathbb{E}[X(s)] \rangle ds \\ &= \int_0^t \langle P(C + \bar{C})\mathbb{E}[X(s)], (C + \bar{C})\mathbb{E}[X(s)] \rangle ds - \mathbb{E} \langle P\{X(t) - \mathbb{E}[X(t)]\}, X(t) - \mathbb{E}[X(t)] \rangle \\ &\leq L \int_0^t \langle (Q + \bar{Q})\mathbb{E}[X(s)], \mathbb{E}[X(s)] \rangle ds. \end{aligned}$$

Consequently,

$$\begin{aligned} & \mathbb{E} \int_0^\infty \left(\langle QX(t), X(t) \rangle + \langle \bar{Q}\mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle \right) dt \\ &= \mathbb{E} \int_0^\infty \left(\langle Q\{X(t) - \mathbb{E}[X(t)]\}, X(t) - \mathbb{E}[X(t)] \rangle + \langle (Q + \bar{Q})\mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle \right) dt \\ &\leq (L + 1) \int_0^\infty \langle (Q + \bar{Q})\mathbb{E}[X(s)], \mathbb{E}[X(s)] \rangle ds < \infty. \end{aligned}$$

This means that the system is $L_{Q, \bar{Q}}^2$ -globally integrable. □

We point out that condition (3.14) holds if (3.9) is true or

$$Q + \bar{Q} > 0.$$

Therefore, to have condition (3.14), we do not have to assume (J)'.

4 MF-Stabilizability

We now return to the controlled linear MF-FSDE (1.1) which is denoted by $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$. With this notation, we see that the uncontrolled MF-FSDE (3.1) is nothing but $[A, \bar{A}, C, \bar{C}; 0, 0, 0, 0]$. Note also that in the case $\bar{A} = \bar{C} = 0$ and $\bar{B} = \bar{D} = 0$, the system is a usual controlled linear SDE, which is simply denoted by $[A, C; B, D] \equiv [A, 0, C, 0; B, 0, D, 0]$. Further, in the case $C = 0$ and $D = 0$, the system is reduced to a classical controlled linear ODE, which is denoted by $[A; B] \equiv [A, 0, 0, 0; B, 0, 0, 0]$. We now introduce the following notion for general state equation (1.1).

Definition 4.1 (i) Let (J) hold. System $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ is said to be $\text{MF-}L^2_{Q, \bar{Q}}$ -*stabilizable* if there exists a pair $(K, \bar{K}) \in \mathbb{R}^{n \times m} \times \mathbb{R}^{n \times m}$ such that for any $x \in \mathbb{R}^n$ if $X^{K, \bar{K}}(\cdot)$ is the solution to the following:

$$\begin{cases} dX^{K, \bar{K}}(t) = \left\{ (A + BK)X^{K, \bar{K}}(t) + [\bar{A} + \bar{B}\bar{K} + B(\bar{K} - K)]\mathbb{E}[X^{K, \bar{K}}(t)] \right\} dt \\ \quad + \left\{ (C + DK)X^{K, \bar{K}}(t) + [\bar{C} + \bar{D}\bar{K} + D(\bar{K} - K)]\mathbb{E}[X^{K, \bar{K}}(t)] \right\} dW(t), \quad t \geq 0, \\ X^{K, \bar{K}}(0) = x, \end{cases}$$

and

$$u^{K, \bar{K}}(t) = K\{X^{K, \bar{K}}(t) - \mathbb{E}[X^{K, \bar{K}}(t)]\} + \bar{K}\mathbb{E}[X^{K, \bar{K}}(t)], \quad t \geq 0, \quad (4.1)$$

then

$$\mathbb{E} \int_0^\infty \left(\langle QX^{K, \bar{K}}(t), X^{K, \bar{K}}(t) \rangle + \langle \bar{Q}\mathbb{E}[X^{K, \bar{K}}(t)], \mathbb{E}[X^{K, \bar{K}}(t)] \rangle + |u^{K, \bar{K}}(t)|^2 \right) dt < \infty. \quad (4.2)$$

In this case, the pair (K, \bar{K}) is called an $\text{MF-}L^2_{Q, \bar{Q}}$ -*stabilizer* of the system. In the case that (4.2) is replaced by the following:

$$\mathbb{E} \int_0^\infty \left(|X^{K, \bar{K}}(t)|^2 + |u^{K, \bar{K}}(t)|^2 \right) dt < \infty,$$

we simply say that the system is $\text{MF-}L^2$ -*stabilizable*, and (K, \bar{K}) is called an $\text{MF-}L^2$ -*stabilizer* of the system.

(ii) Let (J) hold. System $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ is said to be $L^2_{Q, \bar{Q}}$ -*stabilizable* if there exists a $K \in \mathbb{R}^{n \times m}$ such that for any $x \in \mathbb{R}^n$, if $X^K(\cdot)$ is the solution to the following:

$$\begin{cases} dX^K(t) = \left\{ (A + BK)X^K(t) + (\bar{A} + \bar{B}K)\mathbb{E}[X^K(t)] \right\} dt \\ \quad + \left\{ (C + DK)X^K(t) + (\bar{C} + \bar{D}K)\mathbb{E}[X^K(t)] \right\} dW(t), \quad t \geq 0, \\ X^K(0) = x, \end{cases} \quad (4.3)$$

and

$$u^K(t) = KX^K(t), \quad t \geq 0,$$

then

$$\mathbb{E} \int_0^\infty \left(\langle QX^K(t), X^K(t) \rangle + \langle \bar{Q}\mathbb{E}[X^K(t)], \mathbb{E}[X^K(t)] \rangle + |u^K(t)|^2 \right) dt < \infty. \quad (4.4)$$

In this case, K is called an $L^2_{Q, \bar{Q}}$ -*stabilizer* of the system. In the case that $\bar{Q} = 0$, we simply say that the system is L^2_Q -*stabilizable*, and K is called an L^2_Q -*stabilizer*. If (4.4) is replaced by

$$\mathbb{E} \int_0^\infty \left(|X^K(t)|^2 + |u^K(t)|^2 \right) dt < \infty,$$

we further simply say that the system is L^2 -*stabilizable*, and K is called an L^2 -*stabilizer* of the system.

The importance of the notions defined in the above definition is that if (J) holds and $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ is $\text{MF-}L^2_{Q, \bar{Q}}$ -*stabilizable*, then $\mathcal{U}_{ad}[0, \infty)$ is nonempty since $u^{K, \bar{K}}(\cdot)$ defined by (4.1) is in $\mathcal{U}_{ad}[0, \infty)$. In particular, $\mathcal{U}_{ad}[0, \infty)$ is nonempty if the system $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ is $\text{MF-}L^2$ -*stabilizable*.

It is seen that when system $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ is $\text{MF-}L^2_{Q, \bar{Q}}$ -*stabilizable*, then the uncontrolled system $[A + BK, \bar{A} + \bar{B}\bar{K} + B(\bar{K} - K), C + DK, \bar{C} + \bar{D}\bar{K} + D(\bar{K} - K)]$ is $L^2_{Q, \bar{Q}}$ -globally integrable. Also, system $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ is L^2_Q -*stabilizable* if and only if

$$\mathbb{E} \int_0^\infty \left(\langle QX^K(t), X^K(t) \rangle + |u^K(t)|^2 \right) dt < \infty.$$

Moreover, it is clear that the L^2 -stabilizability of system $[A, C; B, D]$ we defined here is the classic stabilizability of the controlled SDE system.

Note that system (1.1) is $L^2_{Q, \bar{Q}}$ -stabilizable (resp. L^2 -stabilizability) if it is MF- $L^2_{Q, \bar{Q}}$ -stabilizable (resp. MF- L^2 -stabilizability) with $K = \bar{K}$. Therefore, the former is a special case of the later. The following example shows that in general, the MF- L^2 -stabilizability does not imply the L^2 -stabilizability.

Example 4.2 Consider the following one-dimensional controlled MF-FSDE:

$$\begin{cases} dX(t) = \{aX(t) + \bar{a}\mathbb{E}[X(t)] + bu(t) + \bar{b}\mathbb{E}[u(t)]\}dt \\ \quad + \{cX(t) + \bar{c}\mathbb{E}[X(t)] + du(t) + \bar{d}\mathbb{E}[u(t)]\}dW(t), & t \geq 0, \\ X(0) = x. \end{cases}$$

Suppose the above system is MF- L^2 -stabilizable. Then, there are $k, \bar{k} \in \mathbb{R}$ such that with

$$u(t) = k\{X(t) - \mathbb{E}[X(t)]\} + \bar{k}\mathbb{E}[X(t)], \quad t \geq 0,$$

the closed-loop system:

$$\begin{aligned} dX(t) = & \{(a + bk)X(t) + [\bar{a} + \bar{b}\bar{k} + b(\bar{k} - k)]\mathbb{E}[X(t)]\}dt \\ & + \{(c + dk)X(t) + [\bar{c} + \bar{d}\bar{k} + d(\bar{k} - k)]\mathbb{E}[X(t)]\}dW(t), \quad t \geq 0, \end{aligned}$$

is L^2 -globally integrable. By Proposition 3.8, this is equivalent to the following:

$$a + \bar{a} + (b + \bar{b})\bar{k} < 0,$$

and either

$$2(a + bk) + (c + dk)^2 < 0,$$

or

$$2(a + bk) + (c + dk)^2 \geq 0, \quad c + \bar{c} + (d + \bar{d})\bar{k} = 0.$$

Let

$$b + \bar{b} = 1, \quad d = 1, \quad \bar{d} = -1, \quad c + \bar{c} \neq 0.$$

Then we need and only need

$$a + \bar{a} + \bar{k} \equiv -\lambda < 0, \quad 2(a + bk) + (c + k)^2 < 0, \quad (4.5)$$

for some $k, \bar{k} \in \mathbb{R}$. The first condition in (4.5) can always be achieved. The second one is equivalent to the following:

$$0 > k^2 + 2(b + c)k + 2a + c^2 = (k + b + c)^2 + 2a + c^2 - (b + c)^2,$$

which is possible if

$$2a + c^2 - (b + c)^2 < 0. \quad (4.6)$$

On the other hand, in order the system to be stabilizable, we need $k = \bar{k}$, and

$$\begin{aligned} 0 > 2(a + b\bar{k}) + (c + \bar{k})^2 &= 2[a - b(a + \bar{a} + \lambda)] + [c - (a + \bar{a} + \lambda)]^2 \\ &= \lambda^2 - 2(a + \bar{a} + b - c)\lambda + (a + \bar{a} - c)^2 + 2[a - b(a + \bar{a})], \end{aligned}$$

for some $\lambda > 0$. This is impossible if, say,

$$c \geq a + \bar{a} + b, \quad a - b(a + \bar{a}) \geq 0. \quad (4.7)$$

It is easy to find cases that (4.6)–(4.7) hold. Hence, we see that MF- L^2 -stabilizability does not imply L^2 -stabilizability, in general.

Now, we present a result concerning the MF- $L^2_{Q,\bar{Q}}$ -stabilizability of system (1.1).

Theorem 4.3 *Let (J) hold.*

(i) *If system (1.1) is MF- $L^2_{Q,\bar{Q}}$ -stabilizable, then the controlled ODE system $[A + \bar{A}; B + \bar{B}]$ is $L^2_{Q+\bar{Q}}$ -stabilizable, i.e., for some $\bar{K} \in \mathbb{R}^{m \times n}$,*

$$\int_0^\infty |(Q + \bar{Q})^{\frac{1}{2}} e^{[A + \bar{A} + (B + \bar{B})\bar{K}]t}|^2 dt < \infty. \quad (4.8)$$

(ii) *Suppose the following holds for some $\bar{K} \in \mathbb{R}^{m \times n}$ satisfying (4.8):*

$$\mathcal{N}(Q + \bar{Q}) \subseteq \mathcal{N}(C + \bar{C}) + (D + \bar{D})\bar{K}. \quad (4.9)$$

Further, suppose the controlled SDE system $[A, C; B, D]$ is L^2_Q -stabilizable. Then the controlled MF-FSDE system $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ is MF- $L^2_{Q,\bar{Q}}$ -stabilizable.

(iii) *Suppose the following holds for some $\bar{K} \in \mathbb{R}^{n \times m}$ satisfying (4.8):*

$$C + \bar{C} + (D + \bar{D})\bar{K} = 0. \quad (4.10)$$

Then the controlled MF-FSDE system $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ is MF- $L^2_{Q,\bar{Q}}$ -stabilizable.

Proof. Under (4.1), the closed-loop system takes form (4.3). According to Proposition 3.9, we know that if (4.3) is $L^2_{Q,\bar{Q}}$ -globally integrable, it is necessary that (4.8) holds, which proves (i). Further, when (4.8) holds, the system (4.3) is $L^2_{Q,\bar{Q}}$ -globally integrable if either the system $[A + BK, C + DK]$ is stable and (4.9) holds, which proves (ii), or (4.10) holds with the same \bar{K} which proves (iii). \square

The above leads to the following corollary.

Corollary 4.4 (i) *If system (1.1) is MF- L^2 -stabilizable, then the controlled ODE system $[A + \bar{A}; B + \bar{B}]$ is stabilizable, i.e., there exists a $\bar{K} \in \mathbb{R}^{n \times m}$ such that*

$$\sigma(A + \bar{A} + (B + \bar{B})\bar{K}) \subseteq \mathbb{C}^-. \quad (4.11)$$

(ii) *Suppose controlled ODE system $[A + \bar{A}; B + \bar{B}]$ is stabilizable, and controlled SDE system $[A, C; B, D]$ is L^2 -stabilizable. Then the controlled MF-FSDE system $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ is MF- L^2 -stabilizable.*

(iii) *Suppose (4.10) holds for some $\bar{K} \in \mathbb{R}^{n \times m}$ satisfying (4.11). Then the controlled MF-FSDE system $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ is MF- L^2 -stabilizable.*

Note that conditions assumed in (ii) of Corollary 4.4 do not involve \bar{C} and \bar{D} . However, condition (4.10) involves both \bar{C} and \bar{D} . We point out that (4.10) means that

$$\mathcal{R}(C + \bar{C}) \subseteq \mathcal{R}(D + \bar{D}). \quad (4.12)$$

In the case that $m < n$, the above could be a big restriction on $C + \bar{C}$ and $D + \bar{D}$. Moreover, we have to find the same $\bar{K} \in \mathbb{R}^{m \times n}$ such that (4.11) and (4.10) hold at the same time. If we let $(D + \bar{D})^+$ be the Moore-Penrose pseudo-inverse of $D + \bar{D}$ ([7]), then the solution of (4.10) is given by

$$\bar{K} = -(D + \bar{D})^+(C + \bar{C}) + [I - (D + \bar{D})^+(D + \bar{D})]\tilde{K},$$

for some $\tilde{K} \in \mathbb{R}^{m \times n}$. Thus, we need

$$\sigma\left(A + \bar{A} + (B + \bar{B})\left\{-(D + \bar{D})^+(C + \bar{C}) + [I - (D + \bar{D})^+(D + \bar{D})]\tilde{K}\right\}\right) \subseteq \mathbb{C}^-,$$

for some $\tilde{K} \in \mathbb{R}^{m \times n}$, which means the ODE system

$$\left[A + \bar{A} - (B + \bar{B})(D + \bar{D})^+(C + \bar{C}); (B + \bar{B})[I - (D + \bar{D})^+(D + \bar{D})] \right] \quad (4.13)$$

is stabilizable. Hence, we obtain the following result.

Proposition 4.5 *Let (4.12) hold. Then $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ is MF- L^2 -stabilizable if ODE system (4.13) is stabilizable, which is the case, if, in particular, $m = n$, $D + \bar{D}$ is invertible, and*

$$\sigma(A + \bar{A} - (B + \bar{B})(D + \bar{D})^{-1}(C + \bar{C})) \subseteq \mathbb{C}^-. \quad (4.14)$$

Condition (4.14) seems that the MF- L^2 -stabilizability of system $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ could be nothing to do with the stabilizability of the controlled linear SDE system $[A, C; B, D]$. However, in the case that $\bar{A} = \bar{C} = 0$ and $\bar{B} = \bar{D} = 0$, we have the following controlled linear SDE:

$$dX(t) = \{AX(t) + Bu(t)\}dt + \{CX(t) + Du(t)\}dW(t), \quad t \geq 0.$$

Suppose $m = n$ and D^{-1} exists. Then condition (4.14) becomes

$$\sigma(A - BD^{-1}C) \subseteq \mathbb{C}^-. \quad (4.15)$$

In this case, if we take

$$u(t) = -D^{-1}CX(t), \quad t \geq 0,$$

then the closed-loop system becomes

$$dX(t) = (A - BD^{-1}C)X(t)dt, \quad t \geq 0,$$

which is stable if (4.15) holds. Interestingly, if we let

$$\bar{u}(t) = -D^{-1}C\mathbb{E}[X(t)], \quad \forall t \geq 0, \quad (4.16)$$

then the closed-loop system reads

$$\begin{cases} dX(t) = \{AX(t) - BD^{-1}C\mathbb{E}[X(t)]\}dt + C\{X(t) - \mathbb{E}[X(t)]\}dW(t), & t \geq 0, \\ X(0) = x. \end{cases}$$

It is not hard to see that the unique solution $X(\cdot)$ of the above is deterministic and given by

$$X(t) = e^{(A - BD^{-1}C)t}x, \quad t \geq 0.$$

Therefore the system is also asymptotically stable under feedback control (4.16).

5 Stochastic LQ Problems

In this section, we study a classic stochastic LQ problem, which will be crucial for Problem (MF-LQ). We consider the following controlled SDE:

$$\begin{cases} dX(t) = \{AX(t) + Bu(t)\}dt + \{CX(t) + Du(t)\}dW(t), & t \geq 0, \\ X(0) = x, \end{cases}$$

and cost functional

$$J^0(x; u(\cdot)) = \mathbb{E} \int_0^\infty \{ \langle QX(t), X(t) \rangle + \langle Ru(t), u(t) \rangle \} dt.$$

Let

$$\begin{cases} \mathcal{X}_{ad}^Q[0, \infty) = \left\{ X(\cdot) \in \mathcal{X}_{loc}[0, \infty) \mid \mathbb{E} \int_0^\infty \langle QX(t), X(t) \rangle dt < \infty \right\}, \\ \mathcal{U}_{ad}^Q[0, \infty) = \left\{ u(\cdot) \in \mathcal{U}[0, \infty) \mid X(x; u(\cdot)) \in \mathcal{X}_{ad}^Q[0, \infty), \quad \forall x \in \mathbb{R}^n \right\}. \end{cases}$$

5.1 A Classic Stochastic LQ Problem

We introduce the following assumptions.

(J)* The matrices $Q \in \mathcal{S}^n$ and $R \in \mathcal{S}^m$ satisfy

$$Q \geq 0, \quad R > 0.$$

(S)* The system $[A, C; B, D]$ is L_Q^2 -stabilizable.

Let us pose the following problem.

Problem (LQ). For any $x \in \mathbb{R}^n$, find a $u_*(\cdot) \in \mathcal{U}_{ad}^Q[0, \infty)$ such that

$$J^0(x; u_*(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}_{ad}^Q[0, \infty)} J^0(x; u(\cdot)) = V^0(x).$$

We have the following result.

Theorem 5.1 *Let (J)* and (S)* hold. Then Problem (LQ) admits a unique optimal control $u_Q(\cdot) \in \mathcal{U}_{ad}^Q[0, \infty)$. Moreover, the following ARE admits a solution $P \geq 0$:*

$$PA + A^T P + C^T P C + Q - (PB + C^T P D)(R + D^T P D)^{-1}(B^T P + D^T P C) = 0,$$

and Γ is an L_Q^2 -stabilizer of $[A, C; B, D]$, where

$$\Gamma = -(R + D^T P D)^{-1}(B^T P + D^T P C). \quad (5.1)$$

Further, the optimal control $u_Q(\cdot)$ is given by

$$u^Q(t) = \Gamma X^Q(t), \quad t \geq 0,$$

with the optimal state process $X^Q(\cdot)$ being the solution of closed-loop system:

$$\begin{cases} dX^Q(t) = (A + B\Gamma)X^Q(t)dt + (C + D\Gamma)X^Q(t)dW(t), & t \geq 0, \\ X^Q(0) = x, \end{cases}$$

and

$$\langle Px, x \rangle = \inf_{u(\cdot) \in \mathcal{U}_{ad}^Q[0, \infty)} J^0(x; u(\cdot)) \equiv V^0(x), \quad \forall x \in \mathbb{R}^n. \quad (5.2)$$

Proof. First of all, it is clear that under (J)* and (S)*, the set $\mathcal{U}_{ad}^Q[0, \infty)$ is nonempty, and $(x, u(\cdot)) \mapsto J^0(x; u(\cdot))$ is a quadratic functional, coercive with respect to $u(\cdot) \in \mathcal{U}_{ad}^Q[0, \infty)$. Thus for any $x \in \mathbb{R}^n$, there exists a unique optimal control $u^Q(\cdot) \in \mathcal{U}_{ad}^Q[0, \infty)$, and the value function $x \mapsto V^0(x)$ must be of form (5.2) for some $P \geq 0$. We now would like to determine P and the optimal pair $(X_*(\cdot), u_*(\cdot))$. To this end, let us introduce

$$J_T^0(x; u(\cdot)) = \mathbb{E} \int_0^T \{ \langle QX(t), X(t) \rangle + \langle Ru(t), u(t) \rangle \} dt, \quad T > 0,$$

where $u(\cdot) \in \mathcal{U}_{loc}[0, \infty)$ and $X(\cdot) = X(\cdot; x, u(\cdot))$. It is standard that under (J)*, there exists a unique $u_T^Q(\cdot) \in \mathcal{U}[0, T]$ such that

$$V_T^0(x) \equiv \inf_{u(\cdot) \in \mathcal{U}[0, T]} J_T^0(x; u(\cdot)) = J_T^0(x; u_T^Q(\cdot)) = \langle P(0; T)x, x \rangle, \quad \forall x \in \mathbb{R}^n,$$

with $P(\cdot; T)$ being the solution to the following differential Riccati equation:

$$\begin{cases} \dot{P}(t; T) + P(t; T)A + A^T P(t; T) + C^T P(t; T)C + Q \\ - [P(t; T)B + C^T P(t; T)D] [R + D^T P(t; T)D]^{-1} [B^T P(t; T) + D^T P(t; T)C] = 0, \quad t \in [0, T], \\ P(T; T) = 0. \end{cases} \quad (5.3)$$

Moreover, the optimal control $u_T(\cdot)$ can be represented as follows:

$$u_T^Q(t) = \Gamma(t; T)X_T^Q(t), \quad t \in [0, T],$$

with

$$\Gamma(t; T) = -[R + D^T P(t; T)D]^{-1} [B^T P(t; T) + D^T P(t; T)C], \quad t \in [0, T],$$

and $X_T^Q(\cdot)$ is the solution to the following closed-loop system:

$$\begin{cases} dX_T^Q(t) = [A + B\Gamma(t; T)]X_T^Q(t)dt + [C + D\Gamma(t; T)]X_T^Q(t)dW(t), \quad t \in [0, T], \\ X_T^Q(0) = x. \end{cases} \quad (5.4)$$

Now, it is clear that

$$J_T^0(x; u(\cdot)) \leq J_{\bar{T}}^0(x; u(\cdot)), \quad \forall u(\cdot) \in \mathcal{U}[0, \bar{T}], \quad 0 \leq T \leq \bar{T} < \infty.$$

Therefore, one has

$$0 \leq P(0; T) \leq P(0; \bar{T}), \quad \forall 0 \leq T \leq \bar{T} < \infty.$$

On the other hand, since

$$\mathcal{U}_{ad}^Q[0, T] \equiv \left\{ u(\cdot) \Big|_{[0, T]} \mid u(\cdot) \in \mathcal{U}_{ad}^Q[0, \infty) \right\} \subseteq \mathcal{U}[0, T],$$

it is true that

$$\begin{aligned} \langle P(0; T)x, x \rangle &\equiv V^0(x) = \inf_{u(\cdot) \in \mathcal{U}[0, T]} J_T^0(x; u(\cdot)) \leq \inf_{u(\cdot) \in \mathcal{U}_{ad}^Q[0, T]} J_T^0(x; u(\cdot)) \\ &\leq \inf_{u(\cdot) \in \mathcal{U}_{ad}^Q[0, \infty)} J^0(x; u(\cdot)) = V^0(x) \equiv \langle Px, x \rangle, \quad \forall x \in \mathbb{R}^n. \end{aligned}$$

Combining the above, we see that

$$0 \leq P(0; T) \leq P(0; \bar{T}) \leq P, \quad \forall 0 \leq T \leq \bar{T} < \infty.$$

This implies that

$$\lim_{T \rightarrow \infty} P(0; T) = \bar{P} \leq P, \quad (5.5)$$

for some $\bar{P}(\cdot) \geq 0$. Now, we introduce the following differential Riccati equation (on $[0, \infty)$):

$$\begin{cases} -\dot{\bar{P}}(s) + \bar{P}(s)A + A^T \bar{P}(s) + C^T \bar{P}(s)C + Q \\ - [\bar{P}(s)B + C^T \bar{P}(s)D] [R + D^T \bar{P}(s)D]^{-1} [B^T \bar{P}(s) + D^T \bar{P}(s)C] = 0, \quad s \geq 0, \\ \bar{P}(0) = 0. \end{cases}$$

For any $T > 0$, let

$$\tilde{P}(t; T) = \bar{P}(T - t), \quad t \in [0, T].$$

Then by the uniqueness, we must have

$$P(t; T) = \tilde{P}(t; T) = \bar{P}(T - t), \quad t \in [0, T].$$

Hence,

$$P(0; T) = \bar{P}(T), \quad T \geq 0.$$

From (5.5), we have

$$\lim_{t \rightarrow \infty} \bar{P}(t) = \bar{P}. \quad (5.6)$$

This $\bar{P} \geq 0$ must be a solution to the algebraic Riccati equation:

$$\bar{P}A + A^T \bar{P} + C^T \bar{P}C - (\bar{P}B + C^T \bar{P}D)(R + D^T \bar{P}D)^{-1}(B^T \bar{P} + D^T \bar{P}C) + Q = 0.$$

Further, from (5.6), one has

$$\lim_{T \rightarrow \infty} P(t; T) = \lim_{T \rightarrow \infty} \bar{P}(T - t) = \bar{P}, \quad t \geq 0.$$

Consequently,

$$\lim_{T \rightarrow \infty} \Gamma(t; T) = -(R + D^T \bar{P}D)^{-1}(B^T \bar{P} + D^T \bar{P}C) \equiv \Gamma_0, \quad \forall t \geq 0. \quad (5.7)$$

Note that (suppressing $(t; T)$)

$$\begin{aligned} & P(A + B\Gamma) + (A + B\Gamma)^T P + (C + D\Gamma)^T P(C + D\Gamma) + \Gamma^T R \Gamma \\ &= P[A - B(R + D^T P D)^{-1}(B^T P + D^T P C)] + [A - B(R + D^T P D)^{-1}(B^T P + D^T P C)]^T P \\ & \quad + [C - D(R + D^T P D)^{-1}(B^T P + D^T P C)]^T P[C - D(R + D^T P D)^{-1}(B^T P + D^T P C)] \\ & \quad + (PB + C P D^T)(R + D^T P D)^{-1} R(R + D^T P D)^{-1}(B^T P + D^T P C) \\ &= PA + A^T P + C^T P C - PB(R + D^T P D)^{-1}(B^T P + D^T P C) \\ & \quad - (PB + C^T P D)(R + D^T P D)^{-1} B^T P - (PB + C^T P D)(R + D^T P D)^{-1} D^T P C \\ & \quad - C^T P D(R + D^T P D)^{-1}(B^T P + D^T P C) \\ & \quad + (PB + C^T P D)(R + D^T P D)^{-1} D^T P D(R + D^T P D)^{-1}(B^T P + D^T P C) \\ & \quad + (PB + C P D^T)(R + D^T P D)^{-1} R(R + D^T P D)^{-1}(B^T P + D^T P C) \\ &= PA + A^T P + C^T P C - (PB + C^T P D)(R + D^T P D)^{-1}(B^T P + D^T P C) \\ & \quad - (PB + C^T P D)(R + D^T P D)^{-1}(B^T P + D^T P C) \\ & \quad + (PB + C P D^T)(R + D^T P D)^{-1}(B^T P + D^T P C) \\ &= PA + A^T P + C^T P C - (PB + C^T P D)(R + D^T P D)^{-1}(B^T P + D^T P C). \end{aligned}$$

Next, we rewrite the differential Riccati equation (5.3) as follows:

$$\begin{cases} \dot{P}(t; T) + P(t; T)[A + B\Gamma(t; T)] + [A + B\Gamma(t; T)]^T P(t; T) \\ \quad + [C + D\Gamma(t; T)]^T P(t; T)[C + D\Gamma(t; T)] + \Gamma(t; T)^T R \Gamma(t; T) + Q = 0, & t \in [0, T], \\ P(T; T) = 0. \end{cases}$$

It is clear that (see (5.4) and (5.7))

$$\lim_{T \rightarrow \infty} X_T^Q(t) = \bar{X}^Q(t), \quad t \geq 0,$$

with $\bar{X}^Q(\cdot)$ being the solution to the following:

$$\begin{cases} d\bar{X}^Q(t) = (A + B\Gamma_0)\bar{X}^Q(t)dt + (C + D\Gamma_0)\bar{X}^Q(t)dW(t), & t \geq 0, \\ \bar{X}^Q(0) = 0. \end{cases}$$

Further,

$$\begin{aligned}
\langle P(0;T)x, x \rangle &= -\mathbb{E} \int_0^T \left\{ \langle \dot{P}(t;T) + P(t;T)[A + B\Gamma(t;T)] + [A(t;T) + B\Gamma(t;T)]^T P(t;T) \right. \\
&\quad \left. + [C + D\Gamma(t;T)]^T P(t;T)[C + D\Gamma(t;T)] \rangle X_T^Q(t), X_T^Q(t) \right\} dt \\
&= \mathbb{E} \int_0^T \langle [Q + \Gamma(t;T)^T R \Gamma(t;T)] X_T^Q(t), X_T^Q(t) \rangle dt \\
&= \mathbb{E} \int_0^T \left(\langle Q X_T^Q(t), X_T^Q(t) \rangle + \langle R \Gamma(t;T) X_T^Q(t), \Gamma(t;T) X_T^Q(t) \rangle \right) dt.
\end{aligned}$$

Thus, by Fatou's Lemma, we obtain (see also (5.5))

$$\langle Px, x \rangle \geq \langle \bar{P}x, x \rangle \geq \mathbb{E} \int_0^\infty \left(\langle Q \bar{X}^Q(t), \bar{X}^Q(t) \rangle + \langle R \Gamma_0 \bar{X}^Q(t), \Gamma_0 \bar{X}^Q(t) \rangle \right) dt \geq V^0(x) = \langle Px, x \rangle,$$

which implies

$$\bar{P} = P, \quad \Gamma_0 = \Gamma,$$

and $\bar{X}^Q(\cdot) \in \mathcal{X}_{ad}^Q[0, \infty)$. Also, Γ defined by (5.1) is an L_Q^2 -stabilizer of $[A, C; B, D]$. This completes the proof. \square

5.2 Stochastic MF-LQ Problem

Having the above, let us now return to Problem (MF-LQ). We introduce the following assumption.

(S) The controlled ODE system $[A + \bar{A}; B + \bar{B}]$ is stabilizable, and the controlled SDE system $[A, C; B, D]$ is L^2 -stabilizable.

From Corollary 4.4, we know that under (J) and (S), the system $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ is MF- L^2 -stabilizable. We point out that it is possible for us to relax (S) in various ways. However, for the simplicity of presentation, we would like to keep the above (S). Let us first present the following result.

Now, we are ready to state and prove the main result of this paper.

Theorem 5.2 *Let (J) and (S) hold. Then Problem (MF-LQ) admits a unique optimal control $u_*(\cdot) \in \mathcal{U}_{ad}[0, \infty)$, and the following AREs:*

$$\begin{cases} PA + A^T P + C^T P C + Q - (PB + C^T P D)(R + D^T P D)^{-1}(B^T P + D^T P C) = 0, \\ \Pi(A + \bar{A}) + (A + \bar{A})^T \Pi + (C + \bar{C})^T P(C + \bar{C}) + Q + \bar{Q} \\ \quad - [\Pi(B + \bar{B}) + (C + \bar{C})^T P(D + \bar{D})][R + \bar{R} + (D + \bar{D})^T P(D + \bar{D})]^{-1} \\ \quad \cdot [(B + \bar{B})^T \Pi + (D + \bar{D})^T P(C + \bar{C})] = 0, \end{cases} \quad (5.8)$$

admits a solution pair $(P, \Pi) \in \bar{S}_+^n \times \bar{S}_+^n$. Define

$$\begin{cases} \Gamma = -(R + D^T P D)^{-1}(B^T P + D^T P C), \\ \bar{\Gamma} = -[R + \bar{R} + (D + \bar{D})^T P(D + \bar{D})]^{-1}[(B + \bar{B})^T \Pi + (D + \bar{D})^T P(C + \bar{C})]. \end{cases}$$

Then $(\Gamma, \bar{\Gamma})$ is an MF- $L_{Q, \bar{Q}}^2$ -stabilizer of the system. If $X_*(\cdot)$ is the solution to the following MF-FSDE:

$$\begin{cases} dX_*(t) = \left\{ (A + B\Gamma)X_*(t) + [\bar{A} + \bar{B}\bar{\Gamma} + B(\bar{\Gamma} - \Gamma)]\mathbb{E}[X_*(t)] \right\} dt \\ \quad + \left\{ (C + D\Gamma)X_*(t) + [\bar{C} + \bar{D}\bar{\Gamma} + D(\bar{\Gamma} - \Gamma)]\mathbb{E}[X_*(t)] \right\} dW(t), \quad t \geq 0, \\ X_*(0) = x, \end{cases}$$

then

$$\inf_{u(\cdot) \in \mathcal{U}_{ad}[0, \infty)} J(x; u(\cdot)) = J(x; u_*(\cdot)) = \langle \Pi x, x \rangle, \quad \forall x \in \mathbb{R}^n, \quad (5.9)$$

with the optimal control $u_*(\cdot) \in \mathcal{U}_{ad}[0, \infty)$ admits the following state feedback representation:

$$u_*(t) = \Gamma \{X_*(t) - \mathbb{E}[X_*(t)]\} + \bar{\Gamma} \mathbb{E}[X_*(t)], \quad t \geq 0.$$

Proof. We know that under (J) and (S), the set $\mathcal{U}_{ad}[0, \infty)$ is nonempty, and convex. For any $(x, u(\cdot)) \in \mathbb{R}^n \times \mathcal{U}_{ad}[0, \infty)$, let $X(\cdot) = X(\cdot; x, u(\cdot)) \in \mathcal{X}[0, \infty)$. Then $J(x; u(\cdot))$ is well-defined and

$$\begin{aligned} J(x; u(\cdot)) &= \mathbb{E} \int_0^\infty \left\{ \langle QX(t), X(t) \rangle + \langle \bar{Q} \mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle + \langle Ru(t), u(t) \rangle + \langle \bar{R} \mathbb{E}[u(t)], \mathbb{E}[u(t)] \rangle \right\} dt \\ &= \mathbb{E} \int_0^\infty \left\{ \langle Q \{X(t) - \mathbb{E}[X(t)]\}, X(t) - \mathbb{E}[X(t)] \rangle + \langle (Q + \bar{Q}) \mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle \right. \\ &\quad \left. + \langle R \{u(t) - \mathbb{E}[u(t)]\}, u(t) - \mathbb{E}[u(t)] \rangle + \langle (R + \bar{R}) \mathbb{E}[u(t)], \mathbb{E}[u(t)] \rangle \right\} dt \\ &\geq \delta \mathbb{E} \int_0^\infty |u(t)|^2 dt, \end{aligned}$$

for some $\delta > 0$. Therefore, under (J) and (S), the map $u(\cdot) \mapsto J(x; u(\cdot))$ is a quadratic and coercive functional on $\mathcal{U}_{ad}[0, \infty)$. Hence, by a standard argument, we see that optimal control $u_*(\cdot) \in \mathcal{U}_{ad}[0, \infty)$ must uniquely exist, and of course, $X_*(\cdot)$ is also unique. By a standard argument, we can show that value function $V(x)$ is of form (5.9) for some $\Pi \in \mathcal{S}^n$, $\Pi \geq 0$.

Now, for any $T > 0$, let

$$J_T(x; u(\cdot)) = \mathbb{E} \int_0^T \left\{ \langle QX(t), X(t) \rangle + \langle \bar{Q} \mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle + \langle Ru(t), u(t) \rangle + \langle \bar{R} \mathbb{E}[u(t)], \mathbb{E}[u(t)] \rangle \right\} dt.$$

We may pose the following problem.

Problem (LQ)_T. For any $x \in \mathbb{R}^n$, find a $u_T(\cdot) \in \mathcal{U}[0, T]$ such that

$$J_T(x; u_T(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[0, T]} J_T(x; u(\cdot)) \equiv V_T(x).$$

By [35], for Problem (LQ)_T, under (J), we have a unique $u_T(\cdot) \in \mathcal{U}[0, T]$ such that

$$V_T(x) = \inf_{u(\cdot) \in \mathcal{U}[0, T]} J_T(x; u(\cdot)) = J_T(x; u_T(\cdot)) = \langle \Pi(0; T)x, x \rangle, \quad \forall x \in \mathbb{R}^n,$$

where

$$\begin{cases} \dot{P}(t; T) + P(t; T)A + A^T P(t; T) + C^T P(t; T)C + Q \\ \quad - [P(t; T)B + C^T P(t; T)D] [R + D^T P(t; T)D]^{-1} [B^T P(t; T) + D^T P(t; T)C] = 0, \quad t \in [0, T], \\ P(T; T) = 0, \end{cases} \quad (5.10)$$

and

$$\begin{cases} \dot{\Pi}(t; T) + \Pi(t; T)(A + \bar{A}) + (A + \bar{A})^T \Pi(t; T) + (C + \bar{C})^T P(t; T)(C + \bar{C}) + Q + \bar{Q} \\ \quad - [\Pi(t; T)(B + \bar{B}) + (C + \bar{C})^T P(t; T)(D + \bar{D})] [R + \bar{R} + (D + \bar{D})^T P(t; T)(D + \bar{D})]^{-1} \\ \quad \cdot [(B + \bar{B})^T \Pi(t; T) + (D + \bar{D})^T P(t; T)(C + \bar{C})] = 0, \quad t \in [0, T], \\ \Pi(T; T) = 0. \end{cases}$$

Further, if we define

$$\begin{cases} \Gamma(t; T) = -[R + D^T P(t; T)D]^{-1} [B^T P(t; T) + D^T P(t; T)C], \\ \bar{\Gamma}(t; T) = -[R + \bar{R} + (D + \bar{D})^T P(t; T)(D + \bar{D})]^{-1} [(B + \bar{B})^T P(t; T) + (D + \bar{D})^T P(t; T)(C + \bar{C})], \end{cases}$$

then the optimal control $u_T(\cdot)$ admits the following state feedback representation:

$$u_T(t) = \Gamma(t; T) \{X_T(t) - \mathbb{E}[X_T(t)]\} + \bar{\Gamma}(t; T) \mathbb{E}[X_T(t)], \quad t \in [0, T],$$

where $X_T(\cdot)$ is the solution to the closed-loop system:

$$\begin{cases} dX_T(t) = \left\{ [A + B\Gamma(t; T)]X_T(t) + [\bar{A} + \bar{B}\bar{\Gamma}(t; T) + B(\bar{\Gamma}(t; T) - \Gamma(t; T))] \mathbb{E}[X_T(t)] \right\} dt \\ \quad + \left\{ [C + D\Gamma(t; T)]X_T(t) + [\bar{C} + \bar{D}\bar{\Gamma}(t; T) + D(\bar{\Gamma}(t; T) - \Gamma(t; T))] \mathbb{E}[X_T(t)] \right\} dW(t), \\ \quad t \in [0, T], \\ X_T(0) = x. \end{cases} \quad (5.11)$$

Observe that (5.10) coincides with (5.3). By the proof of Theorem 5.1, we see that

$$\lim_{T \rightarrow \infty} P(t; T) = P, \quad t \geq 0.$$

Hence,

$$\lim_{T \rightarrow \infty} \Gamma(t; T) = -(R + D^T P D)^{-1} (B^T P + D^T P C) \equiv \Gamma, \quad t \geq 0.$$

Now, we introduce the following differential Riccati equation (on $[0, \infty)$):

$$\begin{cases} -\dot{\bar{\Pi}}(s) + \bar{\Pi}(s)(A + \bar{A}) + (A + \bar{A})^T \bar{\Pi}(s) + (C + \bar{C})^T \bar{P}(s)(C + \bar{C}) + Q + \bar{Q} \\ \quad - [\bar{\Pi}(s)(B + \bar{B}) + (C + \bar{C})^T \bar{P}(s)(D + \bar{D})] [R + \bar{R} + (D + \bar{D})^T \bar{P}(s)(D + \bar{D})]^{-1} \\ \quad \cdot [(B + \bar{B})^T \bar{\Pi}(s) + (D + \bar{D})^T \bar{P}(s)(C + \bar{C})] = 0, \quad t \geq 0, \\ \bar{\Pi}(0) = 0. \end{cases}$$

For any $T > 0$, let

$$\tilde{\Pi}(t; T) = \bar{\Pi}(T - t), \quad t \in [0, T].$$

Then by the uniqueness, we must have

$$\Pi(t; T) = \tilde{\Pi}(t; T) = \bar{\Pi}(T - t), \quad t \in [0, T].$$

Hence,

$$\Pi(0; T) = \bar{\Pi}(T), \quad T \geq 0.$$

Similar to the proof of Theorem 5.1, we have that

$$0 \leq \Pi(0; T) \leq \Pi(0; \bar{T}) \leq \Pi, \quad 0 \leq T \leq \bar{T} < \infty.$$

Thus,

$$\lim_{t \rightarrow \infty} \bar{\Pi}(t) = \bar{\Pi} \leq \Pi.$$

Further, $\bar{\Pi}$ must be a solution to the following ARE:

$$\begin{aligned} & \bar{\Pi}(A + \bar{A}) + (A + \bar{A})^T \bar{\Pi} + (C + \bar{C})^T P(C + \bar{C}) + Q + \bar{Q} \\ & - [\bar{\Pi}(B + \bar{B}) + (C + \bar{C})^T P(D + \bar{D})] [R + \bar{R} + (D + \bar{D})^T P(D + \bar{D})]^{-1} \\ & \cdot [(B + \bar{B})^T \bar{\Pi} + (D + \bar{D})^T P(C + \bar{C})] = 0, \end{aligned}$$

Also,

$$\lim_{T \rightarrow \infty} \Pi(t; T) = \lim_{T \rightarrow \infty} \bar{\Pi}(T - t) = \bar{\Pi}, \quad t \geq 0.$$

Then

$$\lim_{T \rightarrow \infty} \bar{\Gamma}(t; T) = \bar{\Gamma}_0 = -[R + \bar{R} + (D + \bar{D})^T P(D + \bar{D})]^{-1} [(B + \bar{B})^T \bar{\Pi} + (D + \bar{D})^T P(C + \bar{C})], \quad \forall t \geq 0.$$

Recall that $X_T(\cdot)$ satisfies (5.11). Thus, one has

$$\lim_{T \rightarrow \infty} X_T(t; T) = \bar{X}(t), \quad t \geq 0,$$

with $\bar{X}(\cdot)$ being the solution to the following:

$$\begin{cases} d\bar{X}(t) = \left\{ (A + B\Gamma)\bar{X}(t) + [\bar{A} + \bar{B}\bar{\Gamma}_0 + B(\bar{G}_0 - \Gamma)]\mathbb{E}[\bar{X}(t)] \right\} dt \\ \quad + \left\{ (C + D\Gamma)\bar{X}(t) + [\bar{C} + \bar{D}\bar{\Gamma}_0 + D(\bar{\Gamma}_0 - \Gamma)]\mathbb{E}[\bar{X}(t)] \right\} dW(t), \quad t \in [0, T], \\ \bar{X}(0) = x. \end{cases}$$

On the other hand,

$$\begin{aligned} \langle \Pi(0; T)x, x \rangle &= J_T(x; u_T(\cdot)) \\ &= \mathbb{E} \int_0^T \left\{ \langle QX_T(t), X_T(t) \rangle + \langle \bar{Q}\mathbb{E}[X_T(t)], \mathbb{E}[X_T(t)] \rangle \right. \\ &\quad + \langle R(\Gamma(t; T)\{X_T(t) - \mathbb{E}[X_T(t)]\} + \bar{\Gamma}(t; T)\mathbb{E}[X_T(t)]), \Gamma(t; T)\{X_T(t) - \mathbb{E}[X_T(t)]\} + \bar{\Gamma}(t; T)\mathbb{E}[X_T(t)] \rangle \\ &\quad \left. + \langle \bar{R}\bar{\Gamma}(t; T)\mathbb{E}[X_T(t)], \bar{\Gamma}(t; T)\mathbb{E}[X_T(t)] \rangle \right\} dt. \end{aligned}$$

Thus, sending $T \rightarrow \infty$, by Fatou's Lemma, we obtain

$$\begin{aligned} \langle \Pi x, x \rangle &\geq \langle \bar{\Pi}x, x \rangle \geq \mathbb{E} \int_0^\infty \left\{ \langle Q\bar{X}(t), \bar{X}(t) \rangle + \langle \bar{Q}\mathbb{E}[\bar{X}(t)], \mathbb{E}[\bar{X}(t)] \rangle \right. \\ &\quad + \langle R(\Gamma\{\bar{X}(t) - \mathbb{E}[\bar{X}(t)]\} + \bar{\Gamma}_0\mathbb{E}[\bar{X}(t)]), \Gamma\{\bar{X}(t) - \mathbb{E}[\bar{X}(t)]\} + \bar{\Gamma}_0\mathbb{E}[\bar{X}(t)] \rangle \\ &\quad \left. + \langle \bar{R}\bar{\Gamma}_0\mathbb{E}[\bar{X}(t)], \bar{\Gamma}_0\mathbb{E}[\bar{X}(t)] \rangle \right\} dt = J(x; \bar{u}(\cdot)) \geq \langle \Pi x, x \rangle. \end{aligned}$$

Hence,

$$\bar{\Pi} = \Pi, \quad \bar{\Gamma}_0 = \bar{\Gamma},$$

and $(\Gamma, \bar{\Gamma})$ is an MF- $L_{Q, \bar{Q}}^2$ -stabilizer of the system, and $(\bar{X}(\cdot), \bar{u}(\cdot)) = (X_*(\cdot), u_*(\cdot))$ is the optimal pair. \square

6 Optimal MF-LQ Controls Presented via Tackling AREs

6.1 Tackling AREs via LMIs

One of the main ideas of this section is to reformulate the AREs as *linear matrix inequalities* (LMIs, for short). Let us introduce the general notion of LMIs according to [1, 27], and develop it to solve our mean-field LQ problem.

Definition 6.1 Let $F_0, F_1, \dots, F_m \in \mathcal{S}^n$ be given. Inequalities consisting of any combination of the following relations

$$F(x) \triangleq F_0 + \sum_{i=1}^m x_i F_i > 0, \quad \text{or} \quad F(x) \triangleq F_0 + \sum_{i=1}^m x_i F_i \geq 0, \quad (6.1)$$

are called LMIs with respect to the variable $x = (x_1, \dots, x_m)^T \in \mathbb{R}^m$. When the LMI is satisfied by a vector x we say that the LMI is feasible and x is a feasible point.

Next, let us state some facts about general *semi-definite programming* (SDP, for short) problems and their duals.

Definition 6.2 Let $c = (c_1, \dots, c_m)^T \in \mathbb{R}^m$ and $F_0, F_1, \dots, F_m \in \mathcal{S}^n$ be given. The following optimization problem

$$\begin{aligned} \min \quad & c^T x, \\ \text{subject to} \quad & F(x) \equiv F_0 + \sum_{i=1}^m x_i F_i \geq 0, \end{aligned} \quad (6.2)$$

is called a semidefinite programming. Moreover, the dual problem of the SDP (6.2) is defined as

$$\begin{aligned} \max \quad & -\text{Tr}(F_0 Z), \\ \text{subject to} \quad & Z \in \mathcal{S}^n, \quad \text{Tr}(Z F_i) = c_i, \quad i = 1, 2, \dots, m, \quad Z \geq 0. \end{aligned} \quad (6.3)$$

The following basic assumption is imposed throughout this section.

Assumption 6.3 The controlled MF-FSDE system $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ is MF- L^2 -stabilizable.

For notational convenience, we rewrite the AREs (5.8) as follows

$$\mathcal{R}(P, Q, \bar{Q}, R, \bar{R}) = 0, \quad \bar{\mathcal{R}}(P, \Pi, Q, \bar{Q}, R, \bar{R}) = 0, \quad (6.4)$$

where

$$\begin{cases} \mathcal{R}(P, Q, \bar{Q}, R, \bar{R}) \triangleq PA + A^T P + C^T P C - (PB + C^T P D)(R + D^T P D)^{-1}(B^T P + D^T P C) + Q, \\ \bar{\mathcal{R}}(P, \Pi, Q, \bar{Q}, R, \bar{R}) \triangleq \Pi(A + \bar{A}) + (A + \bar{A})^T \Pi + (C + \bar{C})^T P(C + \bar{C}) + Q + \bar{Q} \\ \quad - [\Pi(B + \bar{B}) + (C + \bar{C})^T P(D + \bar{D})][R + \bar{R} + (D + \bar{D})^T P(D + \bar{D})]^{-1} \\ \quad \cdot [(B + \bar{B})^T \Pi + (D + \bar{D})^T P(C + \bar{C})]. \end{cases}$$

Lemma 6.4 Let $Q_1, \bar{Q}_1, Q_2, \bar{Q}_2 \in \mathcal{S}^n$ and $R_1, \bar{R}_1, R_2, \bar{R}_2 \in \mathcal{S}^m$ be given satisfying

$$Q_1 \leq Q_2, \quad \bar{Q}_1 \leq \bar{Q}_2, \quad R_1 \leq R_2, \quad \bar{R}_1 \leq \bar{R}_2.$$

Assume that there exists (P_0, Π_0) such that

$$\mathcal{R}(P_0, Q_1, \bar{Q}_1, R_1, \bar{R}_1) > 0, \quad \bar{\mathcal{R}}(P_0, \Pi_0, Q_1, \bar{Q}_1, R_1, \bar{R}_1) > 0.$$

Then there exist (P_1^*, Π_1^*) and (P_2^*, Π_2^*) satisfying

$$\begin{cases} \mathcal{R}(P_i^*, Q_i, \bar{Q}_i, R_i, \bar{R}_i) = 0, & \bar{\mathcal{R}}(P_i^*, \Pi_i^*, Q_i, \bar{Q}_i, R_i, \bar{R}_i) = 0, & \text{for } i = 1, 2, \\ P_1^* \leq P_2^* & \text{and} & \Pi_1^* \leq \Pi_2^*. \end{cases}$$

Proof. By the assumptions of this Lemma, (P_0, Π_0) must also satisfy

$$\mathcal{R}(P_0, Q_2, \bar{Q}_2, R_2, \bar{R}_2) > 0, \quad \bar{\mathcal{R}}(P_0, \Pi_0, Q_2, \bar{Q}_2, R_2, \bar{R}_2) > 0.$$

It then follows from Proposition A.11 that there exist (P_1^*, Π_1^*) and (P_2^*, Π_2^*) , which are the maximal solutions of their respective AREs:

$$\mathcal{R}(P_i^*, Q_i, \bar{Q}_i, R_i, \bar{R}_i) = 0, \quad \bar{\mathcal{R}}(P_i^*, \Pi_i^*, Q_i, \bar{Q}_i, R_i, \bar{R}_i) = 0, \quad \text{for } i = 1, 2.$$

Furthermore, (P_1^*, Π_1^*) must satisfy

$$\mathcal{R}(P_1^*, Q_2, \bar{Q}_2, R_2, \bar{R}_2) \geq 0, \quad \bar{\mathcal{R}}(P_1^*, \Pi_1^*, Q_2, \bar{Q}_2, R_2, \bar{R}_2) \geq 0.$$

Hence $P_1^* \leq P_2^*$ and $\Pi_1^* \leq \Pi_2^*$ because (P_2^*, Π_2^*) is the maximal solution to its AREs. □

Consider the following SDP problem

$$\begin{aligned} & \max \quad \mathbf{Tr}(P) + \mathbf{Tr}(\Pi), \\ & \text{subject to} \quad \left\{ \begin{aligned} & \left[\begin{array}{c|c} PA + A^T P + C^T P C + Q & PB + C^T P D \\ \hline B^T P + D^T P C & R + D^T P D \end{array} \right] \geq 0, \\ & \left[\begin{array}{c|c} \Pi(A + \bar{A}) + (A + \bar{A})^T \Pi \\ \quad + (C + \bar{C})^T P (C + \bar{C}) + Q + \bar{Q} & \Pi(B + \bar{B}) + (C + \bar{C})^T P (D + \bar{D}) \\ \hline (B + \bar{B})^T \Pi + (D + \bar{D})^T P (C + \bar{C}) & R + \bar{R} + (D + \bar{D})^T P (D + \bar{D}) \end{array} \right] \geq 0. \end{aligned} \right. \end{aligned} \quad (6.5)$$

Theorem 6.5 Let $Q, \bar{Q} \in \mathcal{S}^n$, $R, \bar{R} \in \mathcal{S}^m$ be given. The following are equivalent:

- (i) There exists (P_0, Π_0) such that $\mathcal{R}(P_0, Q, \bar{Q}, R, \bar{R}) \geq 0$ and $\bar{\mathcal{R}}(P_0, \Pi_0, Q, \bar{Q}, R, \bar{R}) \geq 0$.
- (ii) There exists a solution to the AREs (6.4).

Moreover, when (i) or (ii) holds, the AREs (6.4) has a maximal solution (P^*, Π^*) which is the unique optimal solution to the SDP problem (6.5).

Proof. We only need to prove that (i) implies (ii). Let P_0 be given as in (i). For any $\epsilon > 0$ and $\bar{\epsilon} > 0$, we have $\mathcal{R}(P_0, Q + \epsilon I, Q + \bar{\epsilon} I, R, \bar{R}) > 0$ and $\bar{\mathcal{R}}(P_0, \Pi_0, Q + \epsilon I, Q + \bar{\epsilon} I, R, \bar{R}) > 0$. Applying Proposition A.11 and Lemma 6.4, we have that for any positive decreasing sequence $\epsilon_i \rightarrow 0$ and $\bar{\epsilon}_i \rightarrow 0$ there exists a decreasing sequence of symmetric matrices

$$P_{\epsilon_0} \geq \dots \geq P_{\epsilon_i} \geq P_{\epsilon_{i+1}} \geq P_0, \quad \Pi_{\bar{\epsilon}_0} \geq \dots \geq \Pi_{\bar{\epsilon}_i} \geq \Pi_{\bar{\epsilon}_{i+1}} \geq \Pi_0$$

such that $\mathcal{R}(P_{\epsilon_i}, Q + \epsilon_i I, \bar{Q} + \bar{\epsilon}_i I, R, \bar{R}) = 0$ and $\bar{\mathcal{R}}(P_{\epsilon_i}, \Pi_{\bar{\epsilon}_i}, \bar{Q} + \epsilon_i I, Q + \bar{\epsilon}_i I, R, \bar{R}) = 0$. Hence the limit $P^* = \lim_{\epsilon_i \rightarrow 0} P_{\epsilon_i}$ and $\Pi^* = \lim_{\bar{\epsilon}_i \rightarrow 0} \Pi_{\bar{\epsilon}_i}$ exist and satisfy

$$\mathcal{R}(P^*, Q, \bar{Q}, R, \bar{R}) = 0, \quad \bar{\mathcal{R}}(P^*, \Pi^*, Q, \bar{Q}, R, \bar{R}) = 0.$$

In addition, (P^*, Π^*) must be the maximal solution of the AREs due to the arbitrariness of (P_0, Π_0) . By Schur's lemma (Lemma A.1), (P^*, Π^*) is an optimal solution to the problem (6.5) due to its maximality. To prove the uniqueness, let (P_*, Π_*) be any optimal solution to (6.5). Then $\mathbf{Tr}(P^* - P_*) + \mathbf{Tr}(\Pi^* - \Pi_*) = 0$ as both (P^*, Π^*) and (P_*, Π_*) are optimal to (6.5). However, $P^* - P_* \geq 0$ and $\Pi^* - \Pi_* \geq 0$ since (P^*, Π^*) is the maximal solution of (6.5). This yields $P^* - P_* = 0$ and $\Pi^* - \Pi_* = 0$. \square

As an immediate consequence of Theorem 6.5, we have the following result for the standard case $Q, \bar{Q} \geq 0$ and $R, \bar{R} > 0$.

Corollary 6.6 If $Q, \bar{Q} \geq 0$ and $R, \bar{R} > 0$, then the AREs (6.4) admits a maximal solution (P^*, Π^*) with $P^*, \Pi^* \geq 0$ which is also the unique solution to the SDP (6.5). In addition, if $Q, \bar{Q} > 0$ and $R, \bar{R} > 0$, then the maximal solution (P^*, Π^*) with $P^*, \Pi^* > 0$ and the feedback control

$$\begin{aligned} u^*(t) = & -(R + D^T P^* D)^{-1} (B^T P^* + D^T P^* C) (X^*(t) - \mathbb{E}[X^*(t)]) \\ & - (R + \bar{R} + (D + \bar{D})^T P^* (D + \bar{D}))^{-1} [(B + \bar{B})^T \Pi^* + (D + \bar{D})^T P^* (C + \bar{C})] \mathbb{E}[X^*(t)] \end{aligned}$$

is stabilizing for the system (1.1).

Proof. When $Q, \bar{Q} \geq 0$ and $R, \bar{R} > 0$, $(P_0, \Pi_0) = (0, 0)$ satisfies the LMIs

$$\mathcal{R}(P, Q, \bar{Q}, R, \bar{R}) \geq 0, \quad \bar{\mathcal{R}}(P, \Pi, Q, \bar{Q}, R, \bar{R}) \geq 0. \quad (6.6)$$

Hence by Theorems 6.5 the AREs (6.4) admits a maximal solution (P^*, Π^*) . Moreover, by the proof of Theorems 6.5, $P^* \geq P_0 = 0$ and $\Pi^* \geq \Pi_0 = 0$. If in addition $Q, \bar{Q} > 0$ and $R, \bar{R} > 0$, then $(\tilde{P}_0, \tilde{\Pi}_0) = (\delta I, \bar{\delta} I)$ solves (6.6) for a sufficiently small $\delta, \bar{\delta} > 0$. Hence $P^* \geq \tilde{P}_0 = \delta I > 0$ and $\Pi^* \geq \tilde{\Pi}_0 = \bar{\delta} I > 0$. Moreover, by virtue of Proposition A.10, the corresponding feedback control is stabilizing since (6.6) is strictly feasible in this case. \square

6.2 Optimal feedback Control

In this subsection, we show that the value function of Problem MF-LQ can be expressed in terms of the maximal solution to the AREs (6.4). Moreover, if there exists an optimal control of Problem MF-LQ then it is necessarily represented as a feedback via the maximal solution to the AREs.

Theorem 6.7 *Assume that Theorem 6.5-(i) holds. Then Problem (MF-LQ) is well-posed and the value function is given by $V(x) = x^T \Pi^* x$, $\forall x \in \mathbb{R}^n$, where (P^*, Π^*) is the maximal solution to the AREs (6.4).*

Proof. The well-posedness has been shown in Theorem 5.2, which also yields $V(x) = x^T \Pi^* x$.

Now, for any fixed $\epsilon > 0$, the LMIs

$$\mathcal{R}(P, Q + \epsilon I, \bar{Q} + \epsilon I, R, \bar{R}) \geq 0, \quad \bar{\mathcal{R}}(P, \Pi, Q + \epsilon I, \bar{Q} + \epsilon I, R, \bar{R}) \geq 0 \quad (6.7)$$

are strictly feasible. Hence by Proposition A.11, there is a maximal solution, denoted by $(P_\epsilon, \Pi_\epsilon)$, to the corresponding AREs

$$\mathcal{R}(P, Q + \epsilon I, \bar{Q} + \epsilon I, R, \bar{R}) = 0, \quad \bar{\mathcal{R}}(P, \Pi, Q + \epsilon I, \bar{Q} + \epsilon I, R, \bar{R}) = 0.$$

In addition, by Proposition A.10, the feedback control $u_\epsilon(t) = \Gamma_\epsilon(X_\epsilon(t) - \mathbb{E}[X_\epsilon(t)]) + \bar{\Gamma}_\epsilon \mathbb{E}[X_\epsilon(t)]$ is stabilizing, where

$$\begin{cases} \Gamma_\epsilon = -(R + D^T P_\epsilon D)^{-1} (B^T P_\epsilon + D^T P_\epsilon C), \\ \bar{\Gamma}_\epsilon = -(R + \bar{R} + (D + \bar{D})^T P_\epsilon (D + \bar{D}))^{-1} [(B + \bar{B})^T \Pi_\epsilon + (D + \bar{D})^T P_\epsilon (C + \bar{C})]. \end{cases}$$

It is easy to verify that $P_\epsilon, \Pi_\epsilon, \Gamma_\epsilon$ and $\bar{\Gamma}_\epsilon$ satisfy the following equations

$$\begin{cases} (A + B\Gamma_\epsilon)^T P_\epsilon + P_\epsilon (A + B\Gamma_\epsilon) + (C + D\Gamma_\epsilon)^T P_\epsilon (C + D\Gamma_\epsilon) = -Q - \epsilon I - \Gamma_\epsilon^T R \Gamma_\epsilon, \\ (A + \bar{A} + B\bar{\Gamma}_\epsilon + \bar{B}\bar{\Gamma}_\epsilon) \Pi_\epsilon + \Pi_\epsilon (A + \bar{A} + B\bar{\Gamma}_\epsilon + \bar{B}\bar{\Gamma}_\epsilon)^T \\ \quad + (C + \bar{C} + D\bar{\Gamma}_\epsilon + \bar{D}\bar{\Gamma}_\epsilon)^T P_\epsilon (C + \bar{C} + D\bar{\Gamma}_\epsilon + \bar{D}\bar{\Gamma}_\epsilon) = -Q - \bar{Q} - 2\epsilon I - \bar{\Gamma}_\epsilon^T (R + \bar{R}) \bar{\Gamma}_\epsilon. \end{cases} \quad (6.8)$$

Applying Lemma A.4 to $M = P_\epsilon$, $N = \Pi_\epsilon$ and substituting $u_\epsilon(t)$ into (A.1), we have

$$\begin{aligned} & \mathbb{E} \int_0^t \left\{ \langle (Q + \epsilon I) X_\epsilon(s), X_\epsilon(s) \rangle + \langle (\bar{Q} + \epsilon I) \mathbb{E}[X_\epsilon(s)], \mathbb{E}[X_\epsilon(s)] \rangle \right. \\ & \quad \left. + \langle R u_\epsilon(s), u_\epsilon(s) \rangle + \langle \bar{R} \mathbb{E}[u_\epsilon(s)], \mathbb{E}[u_\epsilon(s)] \rangle \right\} ds \\ &= \mathbb{E} \int_0^t \left\{ \langle (Q + \epsilon I) (X_\epsilon(s) - \mathbb{E}[X_\epsilon(s)]), (X_\epsilon(s) - \mathbb{E}[X_\epsilon(s)]) \rangle + \langle (Q + \bar{Q} + 2\epsilon I) \mathbb{E}[X_\epsilon(s)], \mathbb{E}[X_\epsilon(s)] \rangle \right. \\ & \quad \left. + \langle R (u_\epsilon(s) - \mathbb{E}[u_\epsilon(s)]), (u_\epsilon(s) - \mathbb{E}[u_\epsilon(s)]) \rangle + \langle (R + \bar{R}) \mathbb{E}[u_\epsilon(s)], \mathbb{E}[u_\epsilon(s)] \rangle \right\} ds \\ &= \mathbb{E} \int_0^t \left\{ \langle (Q + \epsilon I + \Gamma_\epsilon^T R \Gamma_\epsilon) (X_\epsilon(s) - \mathbb{E}[X_\epsilon(s)]), (X_\epsilon(s) - \mathbb{E}[X_\epsilon(s)]) \rangle \right. \\ & \quad \left. + \langle (Q + \bar{Q} + 2\epsilon I + \bar{\Gamma}_\epsilon^T (R + \bar{R}) \bar{\Gamma}_\epsilon) \mathbb{E}[X_\epsilon(s)], \mathbb{E}[X_\epsilon(s)] \rangle \right\} ds \\ &\equiv \mathbb{E} \int_0^t \left\{ (X_\epsilon(s) - \mathbb{E}[X_\epsilon(s)])^T [(A + B\Gamma_\epsilon)^T P_\epsilon + P_\epsilon (A + B\Gamma_\epsilon) + (C + D\Gamma_\epsilon)^T P_\epsilon (C + D\Gamma_\epsilon)] (X_\epsilon(s) - \mathbb{E}[X_\epsilon(s)]) \right. \\ & \quad \left. + \mathbb{E}[X_\epsilon(s)]^T [(C + \bar{C} + D\bar{\Gamma}_\epsilon + \bar{D}\bar{\Gamma}_\epsilon)^T P_\epsilon (C + \bar{C} + D\bar{\Gamma}_\epsilon + \bar{D}\bar{\Gamma}_\epsilon) \right. \\ & \quad \left. + (A + \bar{A} + B\bar{\Gamma}_\epsilon + \bar{B}\bar{\Gamma}_\epsilon) \Pi_\epsilon + \Pi_\epsilon (A + \bar{A} + B\bar{\Gamma}_\epsilon + \bar{B}\bar{\Gamma}_\epsilon)^T] \mathbb{E}[X_\epsilon(s)] \right\} ds \\ &= -\mathbb{E} \left[(X_\epsilon(t) - \mathbb{E}[X_\epsilon(t)])^T P_\epsilon (X_\epsilon(t) - \mathbb{E}[X_\epsilon(t)]) \right] + x^T \Pi_\epsilon x - \mathbb{E}[X_\epsilon(t)]^T \Pi_\epsilon \mathbb{E}[X_\epsilon(t)]. \end{aligned}$$

Since $\lim_{t \rightarrow +\infty} \mathbb{E}[(X_\epsilon(t) - \mathbb{E}[X_\epsilon(t)])^T P_\epsilon (X_\epsilon(t) - \mathbb{E}[X_\epsilon(t)])] = 0$ and $\lim_{t \rightarrow +\infty} \mathbb{E}[X_\epsilon(t)]^T \Pi_\epsilon \mathbb{E}[X_\epsilon(t)] = 0$, we obtain

$$\begin{aligned} x^T \Pi_\epsilon x &= \mathbb{E} \int_0^\infty \left\{ \langle (Q + \epsilon I) X_\epsilon(s), X_\epsilon(s) \rangle + \langle (\bar{Q} + \epsilon I) \mathbb{E}[X_\epsilon(s)], \mathbb{E}[X_\epsilon(s)] \rangle \right. \\ & \quad \left. + \langle R u_\epsilon(s), u_\epsilon(s) \rangle + \langle \bar{R} \mathbb{E}[u_\epsilon(s)], \mathbb{E}[u_\epsilon(s)] \rangle \right\} ds \geq V(x). \end{aligned}$$

On the other hand, since $P^* = \lim_{\epsilon \rightarrow 0} P_\epsilon$ and $\Pi^* = \lim_{\epsilon \rightarrow 0} \Pi_\epsilon$ (similar to the proof of Theorem 6.5), we have $V(x) \leq x^T \Pi^* x$. This completes the proof. \square

Corollary 6.8 *Assume that Theorem 6.5-(i) holds. If there exists an optimal control of Problem (MF-LQ), then it must be unique and represented by the state feedback control*

$$u^*(t) = \Gamma^*(X^*(t) - \mathbb{E}[X^*(t)]) + \bar{\Gamma}^* \mathbb{E}[X^*(t)],$$

where (P^*, Π^*) is the maximal solution to the AREs (6.4), and

$$\begin{cases} \Gamma^* = -(R + D^T P^* D)^{-1} (B^T P^* + D^T P^* C), \\ \bar{\Gamma}^* = -(R + \bar{R} + (D + \bar{D})^T P^* (D + \bar{D}))^{-1} [(B + \bar{B})^T \Pi^* + (D + \bar{D})^T P^* (C + \bar{C})]. \end{cases}$$

Proof. Let $(X^*(\cdot), u^*(\cdot))$ be an optimal pair of the LQ problem. Then a completion of squares shows

$$\begin{aligned} & \mathbb{E} \int_0^t \left\{ \langle Q X^*(s), X^*(s) \rangle + \langle \bar{Q} \mathbb{E}[X^*(s)], \mathbb{E}[X^*(s)] \rangle + \langle R u^*(s), u^*(s) \rangle + \langle \bar{R} \mathbb{E}[u^*(s)], \mathbb{E}[u^*(s)] \rangle \right\} ds \\ &= \mathbb{E} \int_0^t \left\{ \langle Q (X^*(s) - \mathbb{E}[X^*(s)]), (X^*(s) - \mathbb{E}[X^*(s)]) \rangle + \langle (Q + \bar{Q}) \mathbb{E}[X^*(s)], \mathbb{E}[X^*(s)] \rangle \right. \\ & \quad \left. + \langle R (u^*(s) - \mathbb{E}[u^*(s)]), (u^*(s) - \mathbb{E}[u^*(s)]) \rangle + \langle (R + \bar{R}) \mathbb{E}[u^*(s)], \mathbb{E}[u^*(s)] \rangle \right\} ds \\ & \quad - \mathbb{E} \left[(X^*(t) - \mathbb{E}[X^*(t)])^T P^* (X^*(t) - \mathbb{E}[X^*(t)]) \right] + x^T \Pi^* x - \mathbb{E}[X^*(t)]^T \Pi^* \mathbb{E}[X^*(t)] \\ & \quad + \mathbb{E} \int_0^t \left\{ u^*(s) - \mathbb{E}[u^*(s)] - \Gamma^* (X^*(s) - \mathbb{E}[X^*(s)]) \right\}^T (R + D^T P^* D)^{-1} \\ & \quad \cdot \left[u^*(s) - \mathbb{E}[u^*(s)] - \Gamma^* (X^*(s) - \mathbb{E}[X^*(s)]) \right] ds \\ & \quad + \mathbb{E} \int_0^t \left\{ \mathbb{E}[u^*(s)] - \bar{\Gamma}^* \mathbb{E}[X^*(s)] \right\}^T (R + \bar{R} + (D + \bar{D})^T P^* (D + \bar{D}))^{-1} \left[\mathbb{E}[u^*(s)] - \bar{\Gamma}^* \mathbb{E}[X^*(s)] \right] ds. \end{aligned}$$

As $u^*(\cdot)$ is stabilizing, we have

$$\lim_{t \rightarrow +\infty} \mathbb{E} \left[(X^*(t) - \mathbb{E}[X^*(t)])^T P^* (X^*(t) - \mathbb{E}[X^*(t)]) \right] = 0, \quad \lim_{t \rightarrow +\infty} \mathbb{E}[X^*(t)]^T \Pi^* \mathbb{E}[X^*(t)] = 0,$$

which implies

$$\begin{aligned} V(x) &= J(x, u^*(\cdot)) \\ &= x^T \Pi^* x + \mathbb{E} \int_0^\infty \left\{ u^*(s) - \mathbb{E}[u^*(s)] - \Gamma^* (X^*(s) - \mathbb{E}[X^*(s)]) \right\}^T (R + D^T P^* D)^{-1} \\ & \quad \cdot \left[u^*(s) - \mathbb{E}[u^*(s)] - \Gamma^* (X^*(s) - \mathbb{E}[X^*(s)]) \right] ds \\ & \quad + \mathbb{E} \int_0^\infty \left\{ \mathbb{E}[u^*(s)] - \bar{\Gamma}^* \mathbb{E}[X^*(s)] \right\}^T (R + \bar{R} + (D + \bar{D})^T P^* (D + \bar{D}))^{-1} \left[\mathbb{E}[u^*(s)] - \bar{\Gamma}^* \mathbb{E}[X^*(s)] \right] ds. \end{aligned} \tag{6.9}$$

By Theorem 6.7 we have $V(x) = x^T \Pi^* x$. Hence,

$$\begin{cases} \mathbb{E} \int_0^\infty \left\{ u^*(s) - \mathbb{E}[u^*(s)] - \Gamma^* (X^*(s) - \mathbb{E}[X^*(s)]) \right\}^T (R + D^T P^* D)^{-1} \\ \quad \cdot \left[u^*(s) - \mathbb{E}[u^*(s)] - \Gamma^* (X^*(s) - \mathbb{E}[X^*(s)]) \right] ds = 0, \\ \mathbb{E} \int_0^\infty \left\{ \mathbb{E}[u^*(s)] - \bar{\Gamma}^* \mathbb{E}[X^*(s)] \right\}^T (R + \bar{R} + (D + \bar{D})^T P^* (D + \bar{D}))^{-1} \left[\mathbb{E}[u^*(s)] - \bar{\Gamma}^* \mathbb{E}[X^*(s)] \right] ds = 0. \end{cases}$$

As $R + D^T P^* D$ and $R + \bar{R} + (D + \bar{D})^T P^* (D + \bar{D})$ are constant positive definite matrices, $u^*(t)$ has to be in a feedback form $u^*(t) = \Gamma^*(X^*(t) - \mathbb{E}[X^*(t)]) + \bar{\Gamma}^* \mathbb{E}[X^*(t)]$. \square

7 Numerical Examples

In this section, we report our numerical experiments based on the approach developed in the previous sections. Note that the numerical algorithm we have used for checking LMIs or solving SDP [33].

The system dynamics (1.1) in our experiments is specified by the following matrices

$$\begin{aligned}
 A &= \begin{bmatrix} -0.7 & 0.2 & -0.9 & -0.2 & -0.7 \\ 1.0 & -0.6 & 2.0 & -0.6 & -0.8 \\ 0.8 & 0.8 & -1.7 & -1.5 & 1.1 \\ 0.7 & -0.2 & 0.1 & -0.3 & -0.2 \\ 0.6 & -1.0 & -1.3 & 0.6 & -0.2 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} -0.75 & 0.25 & -0.95 & -0.25 & -0.75 \\ 1.05 & -0.65 & 2.05 & -0.65 & -0.85 \\ 0.85 & 0.85 & -1.75 & -1.55 & 1.15 \\ 0.75 & -0.25 & 0.15 & -0.35 & -0.25 \\ 0.65 & -1.05 & -1.35 & 0.65 & -0.25 \end{bmatrix}, \\
 B &= \begin{bmatrix} 1.4 & -0.7 \\ 0.3 & -1.7 \\ 0.1 & -1.7 \\ -0.1 & 0.1 \\ 0.4 & -1.2 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1.45 & -0.75 \\ 0.35 & -1.75 \\ 0.15 & -1.75 \\ -0.15 & 0.15 \\ 0.45 & -1.25 \end{bmatrix}, \\
 C &= \begin{bmatrix} 0.1 & 0.1 & 0.2 & -0.1 & 0.4 \\ -0.1 & -0.3 & 0.2 & -0.1 & -0.3 \\ 0.6 & 0.4 & -0.3 & 0.1 & -0.2 \\ -0.1 & 0.2 & -0.2 & -0.1 & 0.1 \\ -0.2 & 0.2 & 0.3 & 0.2 & -0.3 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 0.15 & 0.15 & 0.25 & -0.15 & 0.45 \\ -0.15 & -0.35 & 0.25 & -0.15 & -0.35 \\ 0.65 & 0.45 & -0.35 & 0.15 & -0.25 \\ -0.15 & 0.25 & -0.25 & -0.15 & 0.15 \\ -0.25 & 0.25 & 0.35 & 0.25 & -0.35 \end{bmatrix}, \\
 D &= \begin{bmatrix} 0.7 & -0.3 \\ 0.2 & -0.8 \\ 0.1 & -0.8 \\ -0.1 & 0.5 \\ 0.2 & -0.6 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 0.75 & -0.35 \\ 0.25 & -0.85 \\ 0.15 & -0.85 \\ -0.15 & 0.55 \\ 0.25 & -0.65 \end{bmatrix}.
 \end{aligned}$$

7.1 Numerical test of MF-L² stabilizability

Since we have shown that the controlled MF-FSDE system is MF-L²-stabilizable in Proposition A.5 if and only if (A.6) is feasible (with respect to the variables \mathbb{X} , $\bar{\mathbb{X}}$, Y and \bar{Y}), we should check the MF-L² stabilizability first by tackling inequalities. After running the calculation of SDP program via Matlab software, the obtained feasible matrices \mathbb{X} , $\bar{\mathbb{X}}$, Y and \bar{Y} satisfy Proposition A.5:

$$\begin{aligned}
 \mathbb{X} &= \begin{bmatrix} 26.1032 & 0.6379 & -7.9410 & 1.4143 & -7.4032 \\ 0.6379 & 17.0911 & -0.4114 & 8.2578 & 1.3415 \\ -7.9410 & -0.4114 & 19.4946 & 1.1492 & 14.0620 \\ 1.4143 & 8.2578 & 1.1492 & 21.8509 & 7.8151 \\ -7.4032 & 1.3415 & 14.0620 & 7.8151 & 40.5193 \end{bmatrix}, \\
 \bar{\mathbb{X}} &= \begin{bmatrix} 0.0471 & -0.0617 & 0.0114 & -0.2361 & -0.0333 \\ -0.0617 & -0.1398 & -0.1104 & 0.2431 & 0.3623 \\ 0.0114 & -0.1104 & 0.0283 & 0.1159 & 0.0443 \\ -0.2361 & 0.2431 & 0.1159 & 0.4583 & 0.0880 \\ -0.0333 & 0.3623 & 0.0443 & 0.0880 & 0.0952 \end{bmatrix}, \\
 Y &= \begin{bmatrix} -12.1167 & -1.8513 & 7.0876 & -11.3987 & -1.6418 \\ 0.9756 & 2.1581 & 5.2614 & -16.0940 & -12.8827 \end{bmatrix}
 \end{aligned}$$

and

$$\bar{Y} = \begin{bmatrix} -0.3539 & -0.0281 & -0.0278 & -0.2997 & 0.1924 \\ -0.0070 & -0.0900 & 0.1334 & -0.4658 & 0.1065 \end{bmatrix}$$

which give rise to the stabilizing feedback control law $u(t) = K(X(t) - \mathbb{E}[X(t)]) + \bar{K}\mathbb{E}[X(t)]$ with the following feedback gain

$$K = Y\mathbb{X}^{-1} \begin{bmatrix} -0.3725 & 0.1843 & 0.3405 & -0.5390 & -0.1289 \\ 0.1689 & 0.5864 & 0.6700 & -0.8716 & -0.3709 \end{bmatrix}$$

and

$$\bar{K} = \bar{Y}\bar{\mathbb{X}}^{-1} = \begin{bmatrix} 4.0644 & 1.2449 & -3.7655 & 1.9996 & -1.3936 \\ 4.2782 & 0.3405 & 0.6593 & 0.7858 & 0.2849 \end{bmatrix}.$$

7.2 Numerical solutions of SARE

Now we tackle the SARE (6.4) for the following Q , \bar{Q} , R and \bar{R} via solving the SDP problem (6.5):

$$Q = \text{diag}([0, 1, 1, 0, 1]) \quad \text{and} \quad \bar{Q} = \text{diag}([0, 0.5, 1, 0, 0.5]),$$

and

$$R = \text{diag}([1, 1]) \quad \text{and} \quad \bar{R} = \text{diag}([1.5, 1]).$$

We then gain the following solution (P, Π)

$$P = \begin{bmatrix} 0.4151 & 0.3890 & 0.2068 & 0.0162 & -0.4059 \\ 0.3890 & 2.7208 & 1.9097 & -2.6074 & -0.7756 \\ 0.2068 & 1.9097 & 1.8535 & -1.8330 & -0.8979 \\ 0.0162 & -2.6074 & -1.8330 & 4.2403 & -0.2665 \\ -0.4059 & -0.7756 & -0.8979 & -0.2665 & 2.1537 \end{bmatrix}$$

and

$$\Pi = \begin{bmatrix} 0.6147 & 0.5721 & 0.2644 & -0.1455 & -0.6138 \\ 0.5721 & 4.2579 & 2.8706 & -4.4158 & -0.6536 \\ 0.2644 & 2.8706 & 2.6758 & -2.6653 & -1.0890 \\ -0.1455 & -4.4158 & -2.6653 & 6.8158 & -1.0674 \\ -0.6138 & -0.6536 & -1.0890 & -1.0674 & 3.1641 \end{bmatrix}.$$

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A Appendix

A.1 Some useful lemmas

The well-known Schur lemma in [10] plays a key technical role.

Lemma A.1 (Schur's lemma) *Let matrices $M = M^T$, N and $R = R^T > 0$ be given with appropriate dimensions. Then the following conditions are equivalent:*

- (i) $M - NR^{-1}N^T \geq$ (resp. $>$) 0 .
- (ii) $\begin{bmatrix} M & N \\ N^T & R \end{bmatrix} \geq$ (resp. $>$) 0 .
- (iii) $\begin{bmatrix} R & N^T \\ N & M \end{bmatrix} \geq$ (resp. $>$) 0 .

In the original Schur lemma, the matrix R is required to be nonsingular. When R is possibly singular, we have an extended Schur's lemma making use of some generalized inverse matrices. To make it more precise, for any matrix M , there exists a unique matrix M^+ , called the *Moore-Penrose inverse* [31], such that

$$MM^+M = M, \quad M^+MM^+ = M^+, \quad (MM^+)^T = MM^+, \quad (M^+M)^T = M^+M.$$

Lemma A.2 *For a symmetric matrix S , we have*

- (i) $S^+ = (S^+)^T$.
- (ii) $S \geq 0$ if and only if $S^+ \geq 0$.
- (iii) $SS^+ = S^+S$.

Its proof can be found in [1].

Lemma A.3 (Extended Schur's lemma) Let matrices $M = M^T, N$ and $R = R^T$ be given with appropriate dimensions. Then the following conditions are equivalent:

- (i) $M - NR^+N^T \geq 0$, $R \geq 0$, and $N(I - RR^+) = 0$.
- (ii) $\begin{bmatrix} M & N \\ N^T & R \end{bmatrix} \geq 0$.
- (iii) $\begin{bmatrix} R & N^T \\ N & M \end{bmatrix} \geq 0$.

Its proof can be found in [5].

Lemma A.4 Let a constant matrix $M, N \in \mathcal{S}^n$ be given. Then for any admissible pair $(X(\cdot), u(\cdot))$ of the system (1.1), we have

$$\begin{aligned}
& \mathbb{E} \left\{ \int_0^t \left[((X(s) - \mathbb{E}[X(s)])^T (A^T M + M A + C^T M C) (X(s) - \mathbb{E}[X(s)]) \right. \right. \\
& \quad + 2(u(s) - \mathbb{E}[u(s)])^T (B^T M + D^T M C) (X(s) - \mathbb{E}[X(s)]) \\
& \quad + (u(s) - \mathbb{E}[u(s)])^T D^T M D (u(s) - \mathbb{E}[u(s)]) \\
& \quad + ((C + \bar{C})\mathbb{E}[X(s)] + (D + \bar{D})\mathbb{E}[u(s)])^T M ((C + \bar{C})\mathbb{E}[X(s)] + (D + \bar{D})\mathbb{E}[u(s)]) \\
& \quad \left. \left. + \mathbb{E}[X(s)]^T ((A + \bar{A})^T N + N(A + \bar{A})\mathbb{E}[X(s)] + 2\mathbb{E}[X(s)]^T (A + \bar{A})^T N (B + \bar{B})\mathbb{E}[u(s)]) \right] ds \right\} \\
& = \mathbb{E} \left[(X(t) - \mathbb{E}[X(t)])^T M (X(t) - \mathbb{E}[X(t)]) \right] + \mathbb{E}[X(t)]^T N \mathbb{E}[X(t)] - x^T N x, \quad \forall t \geq 0.
\end{aligned} \tag{A.1}$$

Proof. Applying Itô's formula to $(X(t) - \mathbb{E}[X(t)])^T M (X(t) - \mathbb{E}[X(t)])$, integrating from 0 to t , and taking expectations, we easily get the desired result. \square

Proposition A.5 The following assertions are equivalent:

- (i) The controlled MF-FSDE system $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ is MF- L^2 -stabilizable.
- (ii) There exist matrices K, \bar{K} and symmetric matrices X, \bar{X} such that

$$\begin{cases} (A + BK)\mathbb{X} + \mathbb{X}(A + BK)^T + (C + DK)\mathbb{X}(C + DK)^T \\ \quad + (C + \bar{C} + D\bar{K} + \bar{D}\bar{K})\bar{\mathbb{X}}(C + \bar{C} + D\bar{K} + \bar{D}\bar{K})^T < 0, \\ (A + \bar{A} + B\bar{K} + \bar{B}\bar{K})\bar{\mathbb{X}} + \bar{\mathbb{X}}(A + \bar{A} + B\bar{K} + \bar{B}\bar{K})^T < 0, \quad \mathbb{X} > 0, \bar{\mathbb{X}} > 0. \end{cases} \tag{A.2}$$

In this case the feedback $u(t) = K(X(t) - \mathbb{E}[X(t)]) + \bar{K}\mathbb{E}[X(t)]$ is stabilizing.

- (iii) There exist matrices K, \bar{K} and symmetric matrices $\mathbb{X}, \bar{\mathbb{X}}$ such that

$$\begin{cases} (A + BK)^T \mathbb{X} + \mathbb{X}(A + BK) + (C + DK)^T \mathbb{X}(C + DK) \\ \quad + (C + \bar{C} + D\bar{K} + \bar{D}\bar{K})^T \bar{\mathbb{X}}(C + \bar{C} + D\bar{K} + \bar{D}\bar{K}) < 0, \\ (A + \bar{A} + B\bar{K} + \bar{B}\bar{K})\bar{\mathbb{X}} + \bar{\mathbb{X}}(A + \bar{A} + B\bar{K} + \bar{B}\bar{K})^T < 0, \quad \mathbb{X} > 0, \bar{\mathbb{X}} > 0. \end{cases} \tag{A.3}$$

In this case the feedback $u(t) = K(X(t) - \mathbb{E}[X(t)]) + \bar{K}\mathbb{E}[X(t)]$ is stabilizing.

- (iv) There are matrices K, \bar{K} such that for any matrices Y, \bar{Y} there exist unique solution $\mathbb{X}, \bar{\mathbb{X}}$ to the following matrix equations

$$\begin{cases} (A + BK)\mathbb{X} + \mathbb{X}(A + BK)^T + (C + DK)\mathbb{X}(C + DK)^T \\ \quad + (C + \bar{C} + D\bar{K} + \bar{D}\bar{K})\bar{\mathbb{X}}(C + \bar{C} + D\bar{K} + \bar{D}\bar{K})^T + Y = 0, \\ (A + \bar{A} + B\bar{K} + \bar{B}\bar{K})\bar{\mathbb{X}} + \bar{\mathbb{X}}(A + \bar{A} + B\bar{K} + \bar{B}\bar{K})^T + \bar{Y} = 0, \quad \mathbb{X} > 0, \bar{\mathbb{X}} > 0. \end{cases} \quad (\text{A.4})$$

Moreover, if $Y, \bar{Y} > 0$ (resp. $Y, \bar{Y} \geq 0$) then $\mathbb{X}, \bar{\mathbb{X}} > 0$ (resp. $\mathbb{X}, \bar{\mathbb{X}} \geq 0$). Furthermore, in this case the feedback $u(t) = K(X(t) - \mathbb{E}[X(t)]) + \bar{K}\mathbb{E}[X(t)]$ is stabilizing.

- (v) There are matrices K, \bar{K} such that for any matrices Y, \bar{Y} there exist unique solution $\mathbb{X}, \bar{\mathbb{X}}$ to the following matrix equations

$$\begin{cases} (A + BK)^T\mathbb{X} + \mathbb{X}(A + BK) + (C + DK)^T\mathbb{X}(C + DK) \\ \quad + (C + \bar{C} + D\bar{K} + \bar{D}\bar{K})^T\bar{\mathbb{X}}(C + \bar{C} + D\bar{K} + \bar{D}\bar{K}) + Y = 0, \\ (A + \bar{A} + B\bar{K} + \bar{B}\bar{K})\bar{\mathbb{X}} + \bar{\mathbb{X}}(A + \bar{A} + B\bar{K} + \bar{B}\bar{K})^T + \bar{Y} = 0, \quad \mathbb{X} > 0, \bar{\mathbb{X}} > 0. \end{cases} \quad (\text{A.5})$$

Moreover, if $Y, \bar{Y} > 0$ (resp. $Y, \bar{Y} \geq 0$) then $\mathbb{X}, \bar{\mathbb{X}} > 0$ (resp. $\mathbb{X}, \bar{\mathbb{X}} \geq 0$). Furthermore, in this case the feedback $u(t) = K(X(t) - \mathbb{E}[X(t)]) + \bar{K}\mathbb{E}[X(t)]$ is stabilizing.

- (vi) There exist matrices Y, \bar{Y} and symmetric matrices $\mathbb{X}, \bar{\mathbb{X}}$ such that

$$\begin{cases} \left[\begin{array}{c|c} A\mathbb{X} + \mathbb{X}A^T + BY + Y^TB^T & C\mathbb{X} + DY \\ \hline \frac{+(C + \bar{C} + (D + \bar{D})\bar{Y}\bar{\mathbb{X}}^{-1})\bar{\mathbb{X}}(C + \bar{C} + (D + \bar{D})\bar{Y}\bar{\mathbb{X}}^{-1})^T}{\mathbb{X}C^T + Y^TD^T} & -\bar{\mathbb{X}} \end{array} \right] < 0, \\ (A + \bar{A})\bar{\mathbb{X}} + (B + \bar{B})\bar{Y} + \bar{\mathbb{X}}(A + \bar{A})^T + \bar{Y}^T(B + \bar{B})^T < 0, \quad \mathbb{X} > 0, \bar{\mathbb{X}} > 0. \end{cases} \quad (\text{A.6})$$

In this case the feedback $u(t) = Y\mathbb{X}^{-1}(X(t) - \mathbb{E}[X(t)]) + \bar{Y}\bar{\mathbb{X}}^{-1}\mathbb{E}[X(t)]$ is stabilizing.

Proof. For any $n_u \times n$ matrices K, \bar{K} , define an operator $\Phi, \hat{\Phi} : \mathcal{S}^n \rightarrow \mathcal{S}^n$ by

$$\begin{cases} \Phi(\mathbb{X}, \bar{\mathbb{X}}) = (A + BK)\mathbb{X} + \mathbb{X}(A + BK)^T + (C + DK)\mathbb{X}(C + DK)^T \\ \quad + (C + \bar{C} + D\bar{K} + \bar{D}\bar{K})\bar{\mathbb{X}}(C + \bar{C} + D\bar{K} + \bar{D}\bar{K})^T, \\ \hat{\Phi}(\mathbb{X}, \bar{\mathbb{X}}) = (A + \bar{A} + B\bar{K} + \bar{B}\bar{K})\bar{\mathbb{X}} + \bar{\mathbb{X}}(A + \bar{A} + B\bar{K} + \bar{B}\bar{K})^T. \end{cases}$$

If $X(\cdot)$ satisfies the equation (1.1) with the feedback control $u(t) = K(X(t) - \mathbb{E}[X(t)]) + \bar{K}\mathbb{E}[X(t)]$, then by Itô's formula $\mathbb{X}(t) = \mathbb{E}[(X(t) - \mathbb{E}[X(t)])(X(t) - \mathbb{E}[X(t)])^T]$ and $\bar{\mathbb{X}}(t) = \mathbb{E}[X(t)]\mathbb{E}[X(t)]^T$ satisfy the differential matrix systems

$$\frac{d}{dt}\mathbb{X}(t) = \Phi(\mathbb{X}(t), \bar{\mathbb{X}}(t)) \quad \text{and} \quad \frac{d}{dt}\bar{\mathbb{X}}(t) = \hat{\Phi}(\mathbb{X}(t), \bar{\mathbb{X}}(t)).$$

Applying the general result given in the appendix of [19], we have the equivalence between the mean-square stabilizability and each of the assertions (ii)-(v). Furthermore, with $Y = K\mathbb{X}$ and $\bar{Y} = \bar{K}\bar{\mathbb{X}}$ the condition (A.3) is equivalent to

$$\begin{cases} A\mathbb{X} + \mathbb{X}A^T + BY + Y^TB^T + (C\mathbb{X} + DY)\mathbb{X}^{-1}(C\mathbb{X} + DY) \\ \quad + (C + \bar{C} + (D + \bar{D})\bar{Y}\bar{\mathbb{X}}^{-1})\bar{\mathbb{X}}(C + \bar{C} + (D + \bar{D})\bar{Y}\bar{\mathbb{X}}^{-1})^T < 0, \\ (A + \bar{A})\bar{\mathbb{X}} + (B + \bar{B})\bar{Y} + \bar{\mathbb{X}}(A + \bar{A})^T + \bar{Y}^T(B + \bar{B})^T < 0, \quad \mathbb{X} > 0, \bar{\mathbb{X}} > 0. \end{cases}$$

Applying Schur's lemma (Lemma A.1) we have the equivalence of the assertion (vi). \square

Let p^* denote the infimum value of the primal SDP (6.2) and d^* the supremum value of its dual (6.3). Then we have the following results ([33, 1]).

Proposition A.6 $p^* = d^*$ if either of the following conditions holds:

- (i) The primal problem (6.2) is strictly feasible, i.e., there exists an x such that $F(x) > 0$.
- (ii) The dual problem (6.3) is strictly feasible, i.e., there exists a $Z \in \mathcal{S}^n$ with $Z > 0$ and $\mathbf{Tr}(ZF_i) = c_i$, $i = 1, \dots, m$.

If both conditions (i) and (ii) hold, then the optimal sets of both the primal and the dual are nonempty. In this case, the following complementary slackness condition

$$F(x)Z = 0 \quad (\text{A.7})$$

is necessary and sufficient for achieving the optimal values for both problems.

Now we turn to rewrite the AREs (5.8) as

$$\mathcal{R}(P) = 0, \quad \bar{\mathcal{R}}(P, \Pi) = 0, \quad (\text{A.8})$$

where

$$\begin{cases} \mathcal{R}(P) \triangleq PA + A^T P + C^T P C - (PB + C^T P D)(R + D^T P D)^{-1}(B^T P + D^T P C) + Q, \\ \bar{\mathcal{R}}(P, \Pi) \triangleq \Pi(A + \bar{A}) + (A + \bar{A})^T \Pi + (C + \bar{C})^T P(C + \bar{C}) + Q + \bar{Q} - [\Pi(B + \bar{B}) + (C + \bar{C})^T P(D + \bar{D})] \\ \quad \cdot [R + \bar{R} + (D + \bar{D})^T P(D + \bar{D})]^{-1} [(B + \bar{B})^T \Pi + (D + \bar{D})^T P(C + \bar{C})]. \end{cases}$$

In this subsection, we pose an additional assumption that the interior of the set

$$\mathcal{P} = \{(P, \Pi) \in \mathcal{S}^n \times \mathcal{S}^n \mid \mathcal{R}(P) \geq 0, \bar{\mathcal{R}}(P, \Pi) \geq 0\}$$

is nonempty, namely, there exists a $(P_0, \Pi_0) \in \mathcal{S}^n \times \mathcal{S}^n$ such that $\mathcal{R}(P_0) > 0$, and $\bar{\mathcal{R}}(P_0, \Pi_0) \geq 0$.

Consider the following SDP problem

$$\begin{aligned} & \max \quad \mathbf{Tr}(P) + \mathbf{Tr}(\Pi), \\ & \text{subject to} \quad \begin{cases} \left[\begin{array}{c|c} \frac{PA + A^T P + C^T P C + Q}{B^T P + D^T P C} & \frac{PB + C^T P D}{R + D^T P D} \end{array} \right] \geq 0, \\ \left[\begin{array}{c|c} \frac{\Pi(A + \bar{A}) + (A + \bar{A})^T \Pi + (C + \bar{C})^T P(C + \bar{C}) + Q + \bar{Q}}{(B + \bar{B})^T \Pi + (D + \bar{D})^T P(C + \bar{C})} & \frac{\Pi(B + \bar{B}) + (C + \bar{C})^T P(D + \bar{D})}{R + \bar{R} + (D + \bar{D})^T P(D + \bar{D})} \end{array} \right] \geq 0, \\ P - P_0 \geq 0, \\ \Pi - \Pi_0 \geq 0. \end{cases} \end{aligned} \quad (\text{A.9})$$

The constraints of SDP (A.9) can be equivalently expressed as a single LMI

$$F(P, \Pi) \triangleq \left[\begin{array}{c|c|c|c} L(P) & 0 & 0 & 0 \\ \hline 0 & \bar{L}(P, \Pi) & 0 & 0 \\ \hline 0 & 0 & P - P_0 & 0 \\ \hline 0 & 0 & 0 & \Pi - \Pi_0 \end{array} \right] \geq 0, \quad (\text{A.10})$$

where

$$\begin{aligned} L(P) &\triangleq \left[\begin{array}{c|c} \frac{PA + A^T P + C^T P C + Q}{B^T P + D^T P C} & \frac{PB + C^T P D}{R + D^T P D} \end{array} \right], \\ \bar{L}(P, \Pi) &\triangleq \left[\begin{array}{c|c} \frac{\Pi(A + \bar{A}) + (A + \bar{A})^T \Pi + (C + \bar{C})^T P(C + \bar{C}) + Q + \bar{Q}}{(B + \bar{B})^T \Pi + (D + \bar{D})^T P(C + \bar{C})} & \frac{\Pi(B + \bar{B}) + (C + \bar{C})^T P(D + \bar{D})}{R + \bar{R} + (D + \bar{D})^T P(D + \bar{D})} \end{array} \right]. \end{aligned}$$

Proposition A.7 *The dual problem of SDP (A.9) can be formulated as follows*

$$\begin{aligned}
& \max \quad -\mathbf{Tr}(QS + WP_0 + \bar{Q}\bar{S} + \bar{W}\Pi_0) - \mathbf{Tr}(RV + \bar{R}\bar{V}), \\
& \text{subject to} \quad \begin{cases} AS + SA^T + BU + U^T B^T + CSC^T + DUC^T + CU^T D^T + DV D^T + (C + \bar{C})\bar{S}(C + \bar{C})^T \\ \quad + (D + \bar{D})\bar{U}(C + \bar{C})^T + (C + \bar{C})\bar{U}^T(D + \bar{D})^T + (D + \bar{D})\bar{V}(D + \bar{D})^T + W + I = 0, \\ (A + \bar{A})\bar{S} + \bar{S}(A + \bar{A})^T + (B + \bar{B})\bar{U} + \bar{U}^T(B + \bar{B})^T + \bar{W} + I = 0, \\ \left[\begin{array}{cc} S & U^T \\ U & V \end{array} \right] \geq 0, \quad \left[\begin{array}{cc} \bar{S} & \bar{U}^T \\ \bar{U} & \bar{V} \end{array} \right] \geq 0, \quad W \geq 0, \quad \bar{W} \geq 0, \end{cases}
\end{aligned} \tag{A.11}$$

where $S, \bar{S}, W, \bar{W} \in \mathcal{S}^n$, $V, \bar{V} \in \mathcal{S}^m$ and $U, \bar{U} \in \mathbb{R}^{m \times n}$.

Proof. The constraints of the general dual problem (6.3) can be formulated equivalently as the constraints of (A.11). To this end, define the dual variable $Z \in \mathcal{S}^{4n+2m}$ for (6.3) as

$$Z = \left[\begin{array}{cc|cc|c} S & U^T & & & \\ U & V & & & \\ \hline & & Y_1^T & Y_2^T & Y_3^T \\ \hline Y_1 & & \bar{S} & \bar{U}^T & \\ & & \bar{U} & \bar{V} & \\ \hline Y_2 & & Y_4 & W & Y_6^T \\ Y_3 & & Y_5 & Y_6 & \bar{W} \end{array} \right] \geq 0.$$

By the general duality relation $\mathbf{Tr}(ZF_i) = c_i, i = 1, \dots, m$ (see (6.3)) it follows that for any $(P, \Pi) \in \mathcal{S}^n \times \mathcal{S}^n$,

$$\mathbf{Tr}([F(P, \Pi) - F(0, 0)]Z) = -\mathbf{Tr}(P) - \mathbf{Tr}(\Pi),$$

which is equivalent to

$$\begin{aligned}
& \mathbf{Tr} \left([AS + SA^T + BU + U^T B^T + CSC^T + DUC^T + CU^T D^T + DV D^T + (C + \bar{C})\bar{S}(C + \bar{C})^T \right. \\
& \quad + (D + \bar{D})\bar{U}(C + \bar{C})^T + (C + \bar{C})\bar{U}^T(D + \bar{D})^T + (D + \bar{D})\bar{V}(D + \bar{D})^T + W + I]P \\
& \quad \left. + [(A + \bar{A})\bar{S} + \bar{S}(A + \bar{A})^T + (B + \bar{B})\bar{U} + \bar{U}^T(B + \bar{B})^T + \bar{W} + I]\Pi \right) = 0.
\end{aligned}$$

This leads to

$$\begin{cases} AS + SA^T + BU + U^T B^T + CSC^T + DUC^T + CU^T D^T + DV D^T + (C + \bar{C})\bar{S}(C + \bar{C})^T \\ \quad + (D + \bar{D})\bar{U}(C + \bar{C})^T + (C + \bar{C})\bar{U}^T(D + \bar{D})^T + (D + \bar{D})\bar{V}(D + \bar{D})^T + W + I = 0, \\ (A + \bar{A})\bar{S} + \bar{S}(A + \bar{A})^T + (B + \bar{B})\bar{U} + \bar{U}^T(B + \bar{B})^T + \bar{W} + I = 0. \end{cases}$$

On the other hand, the objective of the dual problem (6.3) can be formulated as

$$-\mathbf{Tr}(F(0)Z) = -\mathbf{Tr}(QS + WP_0 + \bar{Q}\bar{S} + \bar{W}\Pi_0) - \mathbf{Tr}(RV + \bar{R}\bar{V}).$$

In particular, since the matrix variables Y_1, Y_2, Y_3, Y_4, Y_5 and Y_6 do not play any role in the above formulation, they can be dropped. Hence, the condition $Z \geq 0$ is equivalent to

$$\left[\begin{array}{cc} S & U^T \\ U & V \end{array} \right] \geq 0, \quad \left[\begin{array}{cc} \bar{S} & \bar{U}^T \\ \bar{U} & \bar{V} \end{array} \right] \geq 0, \quad W \geq 0, \quad \bar{W} \geq 0.$$

This completes the proof. \square

We now show that the MF- L^2 -stability can be regarded as a dual concept of SDP optimality.

Proposition A.8 *The dual problem (A.11) is strictly feasible if and only if the controlled MF-FSDE system $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ is MF- L^2 -stabilizable.*

Proof. First, assume that the controlled MF-FSDE system $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ is MF- L^2 -stabilizable by some feedback $u(t) = K(X(t) - \mathbb{E}[X(t)]) + \bar{K}\mathbb{E}[X(t)]$. Let $\bar{W} > 0$ and $\widehat{W} > 0$ be fixed. Then it follows from the assertion (v) of Proposition A.5 that there exists a unique (S, \bar{S}) satisfying

$$\begin{cases} (A + BK)S + S(A + BK)^T + (C + DK)S(C + DK)^T \\ \quad + (C + \bar{C} + D\bar{K} + \bar{D}\bar{K})\bar{S}(C + \bar{C} + D\bar{K} + \bar{D}\bar{K})^T + \widehat{W} + I = 0, \\ (A + \bar{A} + B\bar{K} + \bar{B}\bar{K})\bar{S} + \bar{S}(A + \bar{A} + B\bar{K} + \bar{B}\bar{K})^T + \widehat{W} + I = 0, \quad S > 0, \bar{S} > 0. \end{cases}$$

Set $U = KS$ and $\bar{U} = \bar{K}\bar{S}$. The above relation can then be rewritten as

$$\begin{cases} AS + SA + BU + U^T B^T + CSC^T + DUC^T + CU^T D^T + DUS^{-1}U^T D^T + (C + \bar{C})\bar{S}(C + \bar{C})^T \\ \quad + (D + \bar{D})\bar{U}(C + \bar{C})^T + (C + \bar{C})\bar{U}^T(D + \bar{D})^T + (D + \bar{D})\bar{U}\bar{S}^{-1}\bar{U}^T(D + \bar{D})^T + \widehat{W} + I = 0, \\ (A + \bar{A})\bar{S} + \bar{S}(A + \bar{A})^T + (B + \bar{B})\bar{U} + \bar{U}^T(B + \bar{B})^T + \widehat{W} + I = 0. \end{cases}$$

Let $\epsilon > 0$ and $\bar{\epsilon} > 0$, define $V = \epsilon I + US^{-1}U^T$, $\bar{V} = \bar{\epsilon} I + \bar{U}\bar{S}^{-1}\bar{U}^T$, $W = -\epsilon DD^T - \bar{\epsilon}(D + \bar{D})(D + \bar{D})^T + \widehat{W}$ and $\bar{W} = \widehat{W}$. Then V, \bar{V}, W and \bar{W} satisfy

$$\begin{cases} AS + SA + BU + U^T B^T + CSC^T + DUC^T + CU^T D^T + DV D^T + (C + \bar{C})\bar{S}(C + \bar{C})^T \\ \quad + (D + \bar{D})\bar{U}(C + \bar{C})^T + (C + \bar{C})\bar{U}^T(D + \bar{D})^T + (D + \bar{D})\bar{V}(D + \bar{D})^T + W + I = 0, \\ (A + \bar{A})\bar{S} + \bar{S}(A + \bar{A})^T + (B + \bar{B})\bar{U} + \bar{U}^T(B + \bar{B})^T + \bar{W} + I = 0. \end{cases}$$

Moreover, by Schur's lemma (Lemma A.1) for $\epsilon > 0$ and $\bar{\epsilon} > 0$ sufficiently small we must have

$$\begin{bmatrix} S & U^T \\ U & V \end{bmatrix} \geq 0, \quad \begin{bmatrix} \bar{S} & \bar{U}^T \\ \bar{U} & \bar{V} \end{bmatrix} \geq 0, \quad W \geq 0, \quad \bar{W} \geq 0.$$

Therefore, the dual problem (A.11) is strictly feasible.

Conversely, assume that the dual problem is strictly feasible. Then there exist $S > 0, \bar{S} > 0, U, \bar{U}, V$ and \bar{V} such that

$$\begin{cases} AS + SA + BU + U^T B^T + CSC^T + DUC^T + CU^T D^T + DV D^T + (C + \bar{C})\bar{S}(C + \bar{C})^T \\ \quad + (D + \bar{D})\bar{U}(C + \bar{C})^T + (C + \bar{C})\bar{U}^T(D + \bar{D})^T + (D + \bar{D})\bar{V}(D + \bar{D})^T < 0, \\ (A + \bar{A})\bar{S} + \bar{S}(A + \bar{A})^T + (B + \bar{B})\bar{U} + \bar{U}^T(B + \bar{B})^T < 0. \end{cases}$$

It follows that

$$\begin{cases} AS + SA + BU + U^T B^T + CSC^T + DUC^T + CU^T D^T + DUS^{-1}U^T D^T + (C + \bar{C})\bar{S}(C + \bar{C})^T \\ \quad + (D + \bar{D})\bar{U}(C + \bar{C})^T + (C + \bar{C})\bar{U}^T(D + \bar{D})^T + (D + \bar{D})\bar{U}\bar{S}^{-1}\bar{U}^T(D + \bar{D})^T < 0, \\ (A + \bar{A})\bar{S} + \bar{S}(A + \bar{A})^T + (B + \bar{B})\bar{U} + \bar{U}^T(B + \bar{B})^T < 0. \end{cases}$$

Define $K = US^{-1}$ and $\bar{K} = \bar{U}\bar{S}^{-1}$. The above inequality is equivalent to

$$\begin{cases} (A + BK)S + S(A + BK)^T + (C + DK)S(C + DK)^T \\ \quad + (C + \bar{C} + D\bar{K} + \bar{D}\bar{K})\bar{S}(C + \bar{C} + D\bar{K} + \bar{D}\bar{K})^T < 0, \\ (A + \bar{A} + B\bar{K} + \bar{B}\bar{K})\bar{S} + \bar{S}(A + \bar{A} + B\bar{K} + \bar{B}\bar{K})^T < 0, \quad S > 0, \bar{S} > 0. \end{cases}$$

We conclude that the assertion (iii) of Proposition A.5 is satisfied. Hence, the controlled MF-FSDE system $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ is MF- L^2 -stabilizable. \square

The following result presents the existence of the solution of the AREs (A.8) via the SDP (A.9).

Proposition A.9 *The optimal set of SDP (A.9) is nonempty and any optimal solution (P_*, Π_*) must satisfy the ARE (A.8).*

Proof. Proposition A.8, along with Proposition A.6, yields the non-emptiness of the optimal set. Next, appealing to the complementary slackness condition (A.7) in Proposition A.6, we conclude that any optimal solution (P_*, Π_*) must satisfy

$$\left[\begin{array}{c|c|c|c} L(P) & 0 & 0 & 0 \\ \hline 0 & \bar{L}(P, \Pi) & 0 & 0 \\ \hline 0 & 0 & P - P_0 & 0 \\ \hline 0 & 0 & 0 & \Pi - \Pi_0 \end{array} \right] \left[\begin{array}{c|c|c|c} S & U^T & Y_1^T & Y_2^T & Y_3^T \\ \hline U & V & \bar{S} & \bar{U}^T & Y_4^T & Y_5^T \\ \hline Y_1 & \bar{U} & \bar{V} & Y_4^T & Y_5^T \\ \hline Y_2 & Y_4 & W & Y_6^T \\ \hline Y_3 & Y_5 & Y_6 & \bar{W} \end{array} \right] = 0,$$

where $S, \bar{S}, U, \bar{U}, V, \bar{V}, W$ and \bar{W} are the corresponding optimal dual variables. From the above we can deduce the following conditions

$$(A^T P_* + P_* A + C^T P_* C + Q)S + (P_* B + C^T P_* D)U = 0, \quad (\text{A.12})$$

$$(A^T P_* + P_* A + C^T P_* C + Q)U^T + (P_* B + C^T P_* D)V = 0, \quad (\text{A.13})$$

$$(B^T P_* + D^T P_* C)S + (R + D^T P_* D)U = 0, \quad (\text{A.14})$$

$$(B^T P_* + D^T P_* C)U^T + (R + D^T P_* D)V = 0, \quad (\text{A.15})$$

$$\left(\Pi_*(A + \bar{A}) + (A + \bar{A})^T \Pi_* + (C + \bar{C})^T P_*(C + \bar{C}) + Q + \bar{Q} \right) \bar{S} + \left(\Pi(B + \bar{B}) + (C + \bar{C})^T P_*(D + \bar{D}) \right) \bar{U} = 0, \quad (\text{A.16})$$

$$\left(\Pi_*(A + \bar{A}) + (A + \bar{A})^T \Pi_* + (C + \bar{C})^T P_*(C + \bar{C}) + Q + \bar{Q} \right) \bar{U}^T + \left(\Pi(B + \bar{B}) + (C + \bar{C})^T P_*(D + \bar{D}) \right) \bar{V} = 0, \quad (\text{A.17})$$

$$\left((B + \bar{B})^T \Pi_* + (D + \bar{D})^T P_*(C + \bar{C}) \right) \bar{S} + \left(R + \bar{R} + (D + \bar{D})^T P_*(D + \bar{D}) \right) \bar{U} = 0, \quad (\text{A.18})$$

$$\left((B + \bar{B})^T \Pi_* + (D + \bar{D})^T P_*(C + \bar{C}) \right) \bar{U}^T + \left(R + \bar{R} + (D + \bar{D})^T P_*(D + \bar{D}) \right) \bar{V} = 0, \quad (\text{A.19})$$

$$(P_* - P_0)W = 0, \quad (\text{A.20})$$

$$(\Pi_* - \Pi_0)\bar{W} = 0. \quad (\text{A.21})$$

Hence (A.14) implies that $U = -(R + D^T P_* D)^{-1}(B^T P_* + D^T P_* C)S$. Putting this into equation (A.12) leads to $\mathcal{R}(P_*)S = 0$. A same manipulation of equations (A.13) and (A.15) yields $\mathcal{R}(P_*)U^T = 0$. Similarly, (A.18) implies that $\bar{U} = -(R + \bar{R} + (D + \bar{D})^T P_*(D + \bar{D}))^{-1}[(B + \bar{B})^T \Pi_* + (D + \bar{D})^T P_*(C + \bar{C})]\bar{S}$. Substituting this into equation (A.16) leads to $\bar{\mathcal{R}}(P_*, \Pi_*)\bar{S} = 0$. And a similar same manipulation of equations (A.17) and (A.19) yields $\bar{\mathcal{R}}(P_*, \Pi_*)\bar{U}^T = 0$. Recall that the dual variables $S, \bar{S}, U, \bar{U}, V, \bar{V}, W, \bar{W}$ satisfy the following constraint

$$\begin{aligned} AS + SA + BU + U^T B^T + CSC^T + DUC^T + CU^T D^T + DV D^T + (C + \bar{C})\bar{S}(C + \bar{C})^T \\ + (D + \bar{D})\bar{U}(C + \bar{C})^T + (C + \bar{C})\bar{U}^T(D + \bar{D})^T + (D + \bar{D})\bar{V}(D + \bar{D})^T + W + I = 0. \end{aligned} \quad (\text{A.22})$$

Multiplying both sides of the above by $\mathcal{R}(P_*, \Pi_*)$ we have

$$\begin{aligned} \mathcal{R}(P_*)[CSC^T + DUC^T + CU^T D^T + DV D^T + (C + \bar{C})\bar{S}(C + \bar{C})^T + (D + \bar{D})\bar{U}(C + \bar{C})^T \\ + (C + \bar{C})\bar{U}^T(D + \bar{D})^T + (D + \bar{D})\bar{V}(D + \bar{D})^T + W + I]\mathcal{R}(P_*) = 0. \end{aligned}$$

It follows from $W \geq 0$ that

$$\begin{aligned} \mathcal{R}(P_*)[CSC^T + DUC^T + CU^T D^T + DUS + \bar{U}^T D^T + (C + \bar{C})\bar{S}(C + \bar{C})^T + (D + \bar{D})\bar{U}(C + \bar{C})^T \\ + (C + \bar{C})\bar{U}^T(D + \bar{D})^T + (D + \bar{D})\bar{U}\bar{S} + \bar{U}^T(D + \bar{D})^T]\mathcal{R}(P_*) \leq 0. \end{aligned} \quad (\text{A.23})$$

Since

$$\begin{bmatrix} S & U^T \\ U & T \end{bmatrix} \geq 0, \quad \begin{bmatrix} \bar{S} & \bar{U}^T \\ \bar{U} & \bar{T} \end{bmatrix} \geq 0,$$

it follows from extended Schur's lemma (Lemma A.3) that $V \geq US^+U^T$, $U = USS^+$, $\bar{V} \geq \bar{U}\bar{S}^+\bar{U}^T$ and $\bar{U} = \bar{U}\bar{S}\bar{S}^+$. By virtue of Lemma A.2 we deduce the following

$$\begin{aligned} & CSC^T + DUC^T + CU^TD^T + DUS^+U^TD^T \\ &= CSS^+SC^T + DUS^+SC^T + CSS^+U^TD^T + DUS^+U^TD^T \\ &= (CS + DU)S^+(SC^T + U^TD^T) \geq 0, \end{aligned} \tag{A.24}$$

and

$$\begin{aligned} & (C + \bar{C})\bar{S}(C + \bar{C})^T + (D + \bar{D})\bar{U}(C + \bar{C})^T + (C + \bar{C})\bar{U}^T(D + \bar{D})^T + (D + \bar{D})\bar{U}\bar{S}^+\bar{U}^T(D + \bar{D})^T \\ &= (C + \bar{C})\bar{S}\bar{S}^+\bar{S}(C + \bar{C})^T + (D + \bar{D})\bar{U}\bar{S}^+\bar{S}(C + \bar{C})^T + (C + \bar{C})\bar{S}\bar{S}^+\bar{U}^T(D + \bar{D})^T \\ &\quad + (D + \bar{D})\bar{U}\bar{S}^+\bar{U}^T(D + \bar{D})^T \\ &= \left((C + \bar{C})\bar{S} + (D + \bar{D})\bar{U} \right) \bar{S}^+ \left(\bar{S}(C + \bar{C})^T + \bar{U}^T(D + \bar{D})^T \right) \geq 0. \end{aligned} \tag{A.25}$$

Then it follows from (A.23) that $\mathcal{R}(P_*)\mathcal{R}(P_*) \leq 0$, resulting in $\mathcal{R}(P_*) = 0$.

Recall that the dual variables $S, \bar{S}, T, \bar{T}, U, \bar{U}, W, \bar{W}$ satisfy the following constraint

$$(A + \bar{A})\bar{S} + \bar{S}(A + \bar{A})^T + (B + \bar{B})\bar{U} + \bar{U}^T(B + \bar{B})^T + \bar{W} + I = 0. \tag{A.26}$$

Multiplying both sides of the above by $\bar{\mathcal{R}}(P_*, \Pi_*)$, we have

$$\bar{\mathcal{R}}(P_*, \Pi_*)[\bar{W} + I]\bar{\mathcal{R}}(P_*, \Pi_*) = 0.$$

Since $\bar{W} \geq 0$, we have $\bar{\mathcal{R}}(P_*, \Pi_*) = 0$. □

The following result indicates that any optimal solution of the primal SDP gives rise to an MF- L^2 stabilizing control of the MF-LQ problem. The readers can refer to [8].

Proposition A.10 *Let (P_*, Π_*) be an optimal solution to the primal SDP (A.9). Then the feedback control $u(t) = \Gamma_*(X(t) - \mathbb{E}[X(t)]) + \bar{\Gamma}_*X(t)$ is stabilizing for the system (1.1), where*

$$\begin{cases} \Gamma_* = -(R + D^TP_*D)^{-1}(B^TP_* + D^TP_*C), \\ \bar{\Gamma}_* = -(R + \bar{R} + (D + \bar{D})^TP_*(D + \bar{D}))^{-1}((B + \bar{B})^T\Pi_* + (D + \bar{D})^TP_*(C + \bar{C})). \end{cases}$$

Proof. Let $S, \bar{S}, U, \bar{U}, V, \bar{V}, W$ and \bar{W} be the corresponding optimal dual variables satisfying (A.12)-(A.21). First, we are to show that $S > 0$ and $\bar{S} > 0$. Suppose that $Sx = 0$ and $\bar{S}x = 0, x \in \mathbb{R}^n$. As U and \bar{U} satisfy

$$U = -(R + D^TP_*D)^{-1}(B^TP_* + D^TP_*C)S$$

and

$$\bar{U} = -(R + \bar{R} + (D + \bar{D})^TP_*(D + \bar{D}))^{-1}[(B + \bar{B})^T\Pi_* + (D + \bar{D})^TP_*(C + \bar{C})]$$

(see (A.14) and (A.18)), we also have $Ux = 0$ and $\bar{U}x = 0$. The dual constraint (A.22) then implies

$$\begin{aligned} & x^T [CSC^T + DUC^T + CU^TD^T + DUS^+\bar{U}^TD^T + (C + \bar{C})\bar{S}(C + \bar{C})^T + (D + \bar{D})\bar{U}(C + \bar{C})^T \\ & \quad + (C + \bar{C})\bar{U}^T(D + \bar{D})^T + (D + \bar{D})\bar{U}\bar{S}^+\bar{U}^T(D + \bar{D})^T + W + I]x \leq 0. \end{aligned}$$

The same manipulation as in the proof of Proposition A.9 gives $x = 0$. As $S \geq 0$ and $\bar{S} \geq 0$, we conclude that $S > 0$ and $\bar{S} > 0$. Now, the equalities (A.22) and (A.26) give

$$\begin{cases} AS + SA + BU + U^T B^T + CSC^T + DUC^T + CU^T D^T + DUS^{-1}U^T D^T + (C + \bar{C})\bar{S}(C + \bar{C})^T \\ \quad + (D + \bar{D})\bar{U}(C + \bar{C})^T + (C + \bar{C})\bar{U}^T(D + \bar{D})^T + (D + \bar{D})\bar{U}\bar{S}^{-1}\bar{U}^T(D + \bar{D})^T < 0, \\ (A + \bar{A})\bar{S} + \bar{S}(A + \bar{A})^T + (B + \bar{B})\bar{U} + \bar{U}^T(B + \bar{B})^T < 0, \quad S > 0, \quad \bar{S} > 0, \end{cases}$$

which is equivalent to the mean-square stabilizability condition (iii) of Proposition A.5 with $K = \Gamma_*$ and $\bar{K} = \bar{\Gamma}_*$. \square

Proposition A.11 *There exists a unique optimal solution to the SDP (A.9), which is also the maximal solution to the AREs (A.8).*

Proof. Let (P_*, Π_*) be an optimal solution to the SDP (A.9). Proposition A.9 shows that (P_*, Π_*) solves the AREs (A.8). To show that it is indeed a maximal solution, define

$$\begin{cases} \Gamma_* = -(R + D^T P_* D)^{-1}(B^T P_* + D^T P_* C), \\ \bar{\Gamma}_* = -(R + \bar{R} + (D + \bar{D})^T P_*(D + \bar{D}))^{-1}[(B + \bar{B})^T \Pi_* + (D + \bar{D})^T P_*(C + \bar{C})]. \end{cases}$$

A simple calculation yields

$$\begin{cases} (A + B\Gamma_*)^T P_* + P_*(A + B\Gamma_*) + (C + D\Gamma_*)^T P_*(C + D\Gamma_*) = -Q - \Gamma_*^T R \Gamma_*, \\ (A + \bar{A} + B\bar{\Gamma}_* + \bar{B}\bar{\Gamma}_*)\Pi_* + \Pi_*(A + \bar{A} + B\bar{\Gamma}_* + \bar{B}\bar{\Gamma}_*)^T \\ \quad + (C + \bar{C} + D\bar{\Gamma}_* + \bar{D}\bar{\Gamma}_*)^T P_*(C + \bar{C} + D\bar{\Gamma}_* + \bar{D}\bar{\Gamma}_*) = -Q - \bar{Q} - \bar{\Gamma}_*^T(R + \bar{R})\bar{\Gamma}_*. \end{cases}$$

On the other hand, it follows from Proposition A.10 that $u_*(t) = \Gamma_*(X_*(t) - \mathbb{E}[X_*(t)]) + \bar{\Gamma}_*\mathbb{E}[X_*(t)]$ is a stabilizing control. A proof similar to that of Theorem 6.7 yields that (P_*, Π_*) is the upper bound of the set \mathcal{P} , namely, (P_*, Π_*) is the maximal solution. Finally, the uniqueness of the solution to the SDP (A.9) follows from the maximality. \square